Solutions Manual for

ROBERT B. COOPER'S

Introduction to

QUEUEING

THEORY

Second Edition

by Børge Tilt

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Foreword

As stated by Børge Tilt in his Preface, he has written this solutions manual with the objective of maximal utility for the instructor, which requires that the solutions be presented in a detailed and orderly fashion. As the reader will see, this objective has been easily met.

I should add that Dr. Tilt's contribution goes far beyond merely solving a large number of (often difficult) exercises. He made many helpful suggestions and corrections, not only to the exercises, but to the text as well. There is no question that his efforts have considerably enhanced the value of my book, and for this I am deeply grateful.

Robert B. Cooper
It is my hope that this solutions manual will prove a valuable supplement to Robert B. Cooper's *Introduction to Queueing Theory*, second edition, and that both teachers and students will benefit from its availability. As far as the teacher is concerned, it offers substantial savings of time, if nothing else. Solutions to the exercises have been written from the point of view that to be of maximal usefulness to the teacher they should contain a detailed and orderly exposition of steps in the solution.

The given solutions are based almost exclusively on information in the book, and such knowledge of mathematics that may reasonably be assumed on the part of the student.

On some of the exercises I have profited from discussions with the author. However, any errors or inaccuracies that may exist are solely my responsibility.

Edward Tilt, Copenhagen, Denmark

November 1980
Chapter 1, Exercise 1

'In what ways...'
No comment.

Chapter 1, Exercise 2

'List some applications of the Erlang loss model.'
No comment.

Chapter 1, Exercise 3

'Discuss ways...'
No comment.

Chapter 1, Exercise 4

'Extend the heuristic conservation-of-flow argument...'

In the present case only one-step state transitions are
effected by arrivals or service completions. The conservation-of-flow principle then leads to the conclusion
that in the long run there will be the same number of
transitions $E_j \rightarrow E_{j+1}$ and $E_{j+1} \rightarrow E_j$, per unit time. Also,
under certain conditions, the mean rate of transitions
$E_j \rightarrow E_{j+1}$ is $\lambda_j P_j$ (due to Poisson arrivals), and the mean
rate of transitions $E_{j+1} \rightarrow E_j$ is $(\mu + \tau)^{-1} P_j$, for $j = 0, 1, \ldots, n-1$
and $(\mu + \tau)^{-1} P_s$, for $j = n, n+1, \ldots$ (two for exponential service times).
If these conditions hold, the conservation-of-flow equations
extending equations (1.1) become

$$\lambda_j P_j = \begin{cases} (\mu + \tau)^{-1} P_j, & (j = 0, 1, \ldots, n-1), \\ (\mu + \tau)^{-1} P_s, & (j = n, n+1, \ldots). \end{cases} \tag{*}$$

In the following eq. (*), they are supposed to hold.
(Chap. 1, Ex. 4a)

a) Recurrent solution of (4) results in

\[ P_j = \begin{cases} \frac{a^j}{j!} P_0 & (j = 1, 2, \ldots, s - 1), \\ \frac{a^s}{s!} P_0 & (j = s, s+1, \ldots), \end{cases} \tag{1} \]

where \( a = \lambda r \). Using (1) and \( \sum_{j=0}^{\infty} P_j = 1 \), we find

\[ P_0 = \left( \sum_{k=0}^{s} \frac{a^k}{k!} + \frac{a^s}{s! (1 - a/\lambda)} \right)^{-1}. \tag{2} \]

b) In calculating (2) we set \( \sum_{j=0}^{\infty} e^{\lambda t} = 1/(1 - \lambda t) \). However, this presupposes \( a < \lambda \). If \( a \geq \lambda \), then eq. (2) is incorrect and should be replaced by \( P_0 = 0 \).

The offered load \( a = \lambda r \) equals the number of servers that on the average (in the long run) must be in service in order to dispose of the work load in such a way that customer orders do not pile up infinitely. Thus \( a < \lambda \) is necessary and sufficient for disposal of the work load without infinite delays.

c) \[ C(s, a) = \frac{1}{s} \sum_{j=1}^{s} P_j = P_0 \sum_{j=1}^{s} \frac{a^j}{j!} = \frac{a^s}{s! (1 - a/\lambda)}. \]

By (2),

\[ C(s, a) = \frac{a^s}{\sum_{k=0}^{s} \frac{a^k}{k!} + \frac{a^s}{s! (1 - a/\lambda)}}. \tag{3} \]

d) In general, when \( a > 1 \), \( C(s, a) \) will depend on the order in which waiting customers are selected from the queue. For example, if the customer with the shortest service time is always selected for service, then \( C(s, a) \) will be different than when the converse policy is adopted. It is therefore necessary to specify service order. The usual assumption, making the model analyzable, is that customers are selected without regard to service time required, as in
(Chap. 1, Ex. 4d)

order-of-arrival service. Without this assumption, the conservation-of-flow equations will not hold, even with Poisson arrivals and exponential service times.

Clearly, $p_j = \frac{P_{w_j}}{\sum_{k=0}^{\infty} P_{w_k}} = \frac{P_{s_1}}{C(s,a)}$ for $j = 0, 1, \ldots$. Hence, when $a < s$, by (1), (2) and (5),

$$p_j = (1 - \frac{s}{a})(\frac{a}{s})^j = (1 - \rho)\rho^j \quad (j = 0, 1, \ldots)$$

where $\rho = a/s$.

Thus,

$$\hat{p}_j = (1 - \rho)^j \hat{p}_0 = \hat{p}_j, \quad (j = 0, 1, \ldots)$$

independently of $k$.

Assume $s = 1$. By (1),

$$P_j = a^j P_0 \quad (j = 0, 1, \ldots),$$

where $P_a = (1 + a + a^2 + \ldots)^{-1} = 1/a$, for $a < s = 1$. Since $\rho = a$,

$$P_i = p_i = (1 - \rho)\rho^j \quad (j = 0, 1, \ldots).$$

Finally, by (3),

$$C(j,a) = \frac{a/(1-a)}{1 + a/(1-a)} = a.$$
'Consider the so-called loss-delay system...'

As in Exercise H, we assume that the mean number of transitions $E_k^+E_k^-$ and $E_{k^*}^+E_{k^*}$ will be estimated correctly. That is the case with Poisson arrivals and exponential service times, respectively.

### a
The conservation-of-flow argument leads to

$$\lambda P = \begin{cases} \left(\left(1 + \frac{\lambda t}{\lambda t + s}ight)^{\lambda t + s}\right) P_{i-1} & (j = 0, 1, \ldots, s-i), \\ \left(1 + \frac{\lambda t}{\lambda t + s}\right)^{\lambda t + s} P_i & (j = s, \ldots, s+s). \end{cases}$$

Hence,

$$P = \begin{cases} \left(1 + \frac{\lambda t}{\lambda t + s}\right)^{\lambda t + s} P_0 & (j = s, \ldots, s+s), \\ \left(1 + \frac{\lambda t}{\lambda t + s}\right)^{\lambda t + s} P_0 & (j = 0, 1, \ldots, s-i) \end{cases},$$

where $\lambda = \lambda t$, and, for all $s$, $P_s = (\sum_{k=0}^{s-i} \frac{\lambda t}{\lambda t + s} \sum_{i=0}^{k} (\lambda t)^i)^{-1}$.

### b
$n = 0$:

$$P_0 = \frac{\lambda t}{\lambda t + s} P_0 \quad (j = 0, 1, \ldots, s),$$

$$P_0 = \left(\sum_{k=0}^{s-i} \frac{\lambda t}{\lambda t + s} \right)^{-1}.$$

$$B(s, n) = P_0 = \frac{\lambda t}{\lambda t + s} \sum_{k=0}^{s-i} \frac{\lambda t}{\lambda t + s}.$$

### c
$n = \infty$:

$$P_0 = \begin{cases} \left(1 + \frac{\lambda t}{\lambda t + s}\right)^{\lambda t + s} P_0 & (j = s, \ldots, s+s), \\ \left(1 + \frac{\lambda t}{\lambda t + s}\right)^{\lambda t + s} P_0 & (j = 0, 1, \ldots, s-i) \end{cases},$$

$$P_0 = \left(\sum_{k=0}^{s-i} \frac{\lambda t}{\lambda t + s} \right)^{-1}.$$

$$C(s, n) = \sum_{i=0}^{\infty} \frac{\lambda t}{\lambda t + s} = \frac{\lambda t/s}{\lambda t + s} \sum_{i=0}^{\infty} \frac{1}{\lambda t + s}.$$

### d
When $n < \infty$, there is no restriction on $a$. (The sums involved in the formula for $P_0$ are finite). When $n = \infty$, the restriction $a < \alpha$ as discussed in Exercise H.
Chapter 1, Exercise 6

Consider a queueing model with two servers and one waiting position.

(a) \[ \begin{align*}
\lambda P_0 &= 2 \pi r P_0, \\
\lambda P_1 &= 3 \pi r P_1, \\
p \lambda P_2 &= 2 \pi r P_2.
\end{align*} \]

(b) \[ \begin{align*}
P_0 &= \frac{1}{2} \pi r, \\
P_1 &= \frac{1}{2} \pi r, \\
P_2 &= p \frac{1}{2} \pi r, \\
P_3 &= (1 + a + \frac{a^2}{2} + p \frac{a^3}{3})^{-1}.
\end{align*} \]

(c) \[ \beta = (1 - p) P_1 + P_2 \]

(d) Let \( \lambda = 2 \) and \( r = 1 \), so that \( a = 2 \), and let \( p = 1/2 \). Then, by part (b),

\[ (P_0, P_1, P_2, P_3) = \left( \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4} \right), \]

and by part (c),

\[ \beta = \frac{1}{2}. \]

(e) \[ \begin{align*}
R &= 200 \lambda (P_1 + P_2) + 100 p \lambda P_2 \\
&= 2 \pi r \frac{r}{2}.
\end{align*} \]

(f) \[ \begin{align*}
C &= 0.50 \cdot 2 + 0.25 \cdot \left( 1 \cdot P_1 + 2 \cdot (P_1 + P_2) \right) \\
&= 1 \frac{2}{5} \frac{r}{2}.
\end{align*} \]

Hence,

\[ \text{Profit} = R - C = 2 \pi r - 1 \frac{2}{5} \frac{r}{2} = 1 \frac{3}{5} \frac{r}{2}. \]
Chapter 2, Exercise 1

In the model considered above, suppose that $c$ cost dollars.

\[
E(\frac{c}{t}) = \sum_{y=1}^{\infty} \frac{c}{j} \Pr(Y = j) \cdot \frac{c}{x} \cdot \frac{\lambda x e^{-\lambda x}}{E(X)} [\text{by (15)}] \\
= \frac{c}{E(X)}.
\]

Chapter 2, Exercise 2

Consider a population modeled as a pure birth process... cf. ex. 6.

If $N(t) = 0$, then, initially, $P_t(0) = 1$ for all $t$. Assume the process $N(t)$ is non-negative. For notational convenience, let $n = N(0)$. The differential-difference equations (13) specialize to

\[
\frac{d}{dt} P_j(t) = (\lambda - \mu) P_j(t) - \lambda P_j(t) + \lambda P_{j-1}(t) (j = 1, 2, \ldots; P_0(t) = 0)
\]

with initial conditions (24). $P_0(t) = 1$ and $P_j(t) = 0$ for $j > n.$ Differential-difference equations are given only for $j \leq n$ as evidently $P_j(t) = 0$ for $j > n$.

In the case $n = 1$, we have

\[
\frac{d}{dt} P_j(t) = (\lambda - \mu) P_j(t) - \lambda P_j(t) + \lambda P_{j-1}(t) (j = 1, 2, \ldots; P_0(t) = 0)
\]

with $P_0(t) = 1$, $P_{j-1}(t) = 0$ for $j > 1.$

For $j = 1, \frac{d}{dt} P_1(t) = -\lambda P_1(t)$, so that

\[
P_1(t) = e^{-\lambda t}.
\]

Hence, $\frac{d}{dt} P_j(t) = -\lambda e^{-\lambda t}$. Applying standard methods in the solution of this linear differential equation, we derive

\[
P_j(t) = e^{-\lambda t} (1 - e^{-\lambda t})^{j-1} (j = 1, 2, \ldots).
\]

The general formula, obtained by induction, is

\[
P_j(t) = e^{-\lambda t} (1 - e^{-\lambda t})^{j-1} (j = 1, 2, \ldots).
\]
Consider a birth-and-death process with \( \mu_0 = 0 \) and \( \mu_j > 0 \) when \( j > 0 \).

First assume an initial state \( E_0 \) with \( j \geq k \). Then states \( E_j \) for \( j = 0, 1, \ldots, k-1 \) are impossible. A relabeling of the states \( (j', k-j) \) and an application of the above theorem, followed by a reverse relabeling, results in the stated equilibrium distribution \( \{ P_j \} \).

If, on the other hand, the initial state is \( E_0 \) with \( j < k \), then \( E_k \) will be reached eventually (with probability 1) and an application of the theorem leads, again, to the indicated limiting distribution.

Thus, unconditionally, the equilibrium distribution is as stated for \( S < \infty \). It also follows that \( P_S = 0 \) for \( S = \infty \).

Compound distributions. — cf. Chap. 5, Ex. 5

Given the probability generating function of \( S_N \)

\[
P( S_N = k ) = \sum_{n=0}^{\infty} P(N = n) P( \sum_{j=1}^{n} X_j = k )
\]

The probability generating function of \( S_N \) is

\[
h(z) = \sum_{n=0}^{\infty} P(N = n) \left( \sum_{j=1}^{n} z^j \right) = \sum_{n=0}^{\infty} P(N = n) \sum_{j=1}^{n} z^j = \sum_{n=0}^{\infty} P(N = n) \frac{z^n - 1}{z - 1} = \sum_{n=0}^{\infty} P(N = n) [f(z)]^n - g(f(z)).
\]

Differentiating \( h(z) \) twice,

\[
h'(z) = g'(f(z)) f(z) + g(f(z)) f'(z) \]

Hence,

\[
h''(2) = g''(f(2)) f'(2) + 2 g'(f(2)) f''(2) - g'(f(2)) f''(2).
\]

\[
h'''(3) = g'''(f(3)) f''(3) + 3 g''(f(3)) f'''(3) - 3 g''(f(3)) f''(3).
\]
By (4.5),
\[ E(S_n) = h(0) - g(0) f(0) = E(N) E(X). \]
By (4.5) and (4.8),
\[ V(S_n) = h(0)^2 + h(0) - [h(0)]^2 \]
\[ = g(0) f(0) + g(0) [f(0)]^2 + g(0) f(0) - [g(0)]^2 \]
\[ = g(0) [f(0) + f(0) - [f(0)]^2 + (g(0) + g(0) - [g(0)]^2) f(0)] \]
\[ = E(N) V(X) + V(N) E^2(X) \]

Chapter 2, Exercise 5

Let \( N_1 \) and \( N_2 \) be..." — cf. Ex 25

Under procedure (a) each of the \( N_1 \) balls will be left unmarked
with probability \( x \), so by its definition, \( g(x) \) is the probability
that none of the \( N_1 \) balls is marked. Similarly, \( g(y) \) is the
probability that none of the \( N_2 \) balls is marked. Hence, under
procedure (a), \( g(x) g(y) \) is the probability that none of the
\( N_1 + N_2 \) balls will be marked, provided that a ball placed in
\( N_1 \) is left unmarked with probability \( x \). Under procedure (b) a ball from either batch will be
left unmarked with probability \( x + y \). It follows that the
probability that none of the \( N_0 \) (\( N_1 \) or \( N_2 \)) balls is marked is
\( g(x + y) \). Hence, under procedure (b), \( g(x + y) g(x + y) \) is the
probability that none of the \( N_1 + N_2 \) balls will be marked.

We take equivalence to mean that the probability distribution
of balls in the two cells is the same for both procedures.
If the procedures are equivalent in this sense, then, whatever
\( x \) and \( y \), the probability that no ball is marked must be the
same under both procedures. Thus equivalence implies
\[ g(x) g(y) = g(x + y). \]
Chapter 2, Exercise 6

For the model of Exercise 2, Section 2.2, define... - of Ex 8

(a) For \( n = N(0) = 1 \), we have found \( P_0(t) = 0 \) and

\[
\frac{d}{dt} P_j(t) = (j-1) \lambda P_{j-1}(t) - j \lambda P_j(t) \quad (j = 1, 2, \ldots)
\]

Hence,

\[
\sum_{j=1}^{\infty} \frac{d}{dt} P_j(t) z^j = \sum_{j=1}^{\infty} (j-1) \lambda P_{j-1}(t) z^j - \sum_{j=1}^{\infty} j \lambda P_j(t) z^j
\]

\[
\frac{d}{dt} P_0(t) = \lambda z \sum_{j=1}^{\infty} (j-1) P_{j-1}(t) z^{j-1} - \lambda z \sum_{j=1}^{\infty} j P_j(t) z^j
\]

\[
\frac{d}{dt} P_j(t) = \lambda z \sum_{j=1}^{\infty} P_{j-1}(t) z^{j-1}
\]

\[
= \lambda z (z - 1) \frac{d}{dt} P_j(t)
\]

(b) We shall verify that the above partial differential equation as well as the initial condition \( n = N(0) = 1 \) are satisfied by

\[
P_0(t) = \frac{e^{-\lambda t}}{1 - ze^{-\lambda t}}
\]

Differentiation results in

\[
\frac{\partial}{\partial z} P_0(t) = \frac{e^{-\lambda t}}{(1 - z e^{-\lambda t})^2}
\]

\[
\frac{\partial}{\partial t} P_0(t) = \lambda z (z - 1) \frac{e^{-\lambda t}}{(1 - z e^{-\lambda t})^2}
\]

It is seen that the expression for \( P_0(t) \) satisfies the partial differential equation derived in part (a). The initial condition \( P(0) = 1 \) translates into the requirement \( P_0(0) = \sum_{j=1}^{\infty} P_j(0) z^j = 1 \), which sure enough is met by the proposed expression for \( P_0(t) \).

In Exercise 2, it was found that if \( n = N(0) = 1 \), then

\[
P_j(t) = \begin{cases} 0 & \text{if } j = 0, \\ e^{-\lambda t}(1 - e^{-\lambda t})^{j-1} & \text{if } j = 1, 2, \ldots. \end{cases}
\]

To this probability distribution corresponds the generating function
(Chap. 2, Ex. 6 b)

\[ g(x,t) = \sum_{i=0}^{\infty} \frac{e^{-xt}}{i!} (1-e^{-x})^i q^i = 2xe^{-xt} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}(i-1)!}{i!} q^i \]

Since \( g(x,t) \) is identical to the given \( P(x,t) \) and because of the one-to-one correspondence between distribution and generating function, it is true that \( P(x,t) \) generates the distribution found in Exercise 2.

In the general case, \( N(0) \cdot n \geq 1 \), it was found in Exercise 2 that \( P_n(t) = P(t) - \ldots - P_{n-1}(t) = 0 \) for all \( t \), and

\[ \frac{\partial}{\partial t} P_n(t) = (t - n + 1) P_{n-1}(t) - n P_n(t) \quad (n = 1, 2, \ldots) \]

In the same way as in part (a) for \( n \geq 1 \), we find

\[ \frac{\partial}{\partial x} P_n(x,t) = \lambda P_{n-1}(x,t) \frac{\partial}{\partial t} P_n(x,t) \]

where \( P_n(x,t) \) is the generating function for \( P_n(t) \), given that \( P_0(t) = 1 \). We shall verify that

\[ P_n(x,t) = P^n(x,t) \]

First, we will show that the proposed solution satisfies the above partial differential equation. This follows from

\[ \frac{\partial}{\partial x} P_n(x,t) = n P^{n-1}(x,t) \frac{\partial}{\partial t} P_n(x,t) \]

\[ \frac{\partial}{\partial x} P_n(x,t) = n P^{n-1}(x,t) \frac{\partial}{\partial t} P_n(x,t) \]

and the previously verified result \( \frac{\partial}{\partial t} P_0(x,t) = \lambda (1-x) \frac{\partial}{\partial x} P_0(x,t) \).

The proposed solution also meets the initial condition. For \( t = 0 \), \( P_n(x,0) = P^n(x,0) = x^n \). By definition, \( P_0(x,0) = P^n(0)^x = P^0(0)^x = \ldots = P^n(x,0) \), equating the coefficients of the two polynomials we see that \( P^n(0) = 1 \) and \( P^n(x,0) = 0 \) for \( 1 \neq x \) as required.

We conclude that \( P_n(x,t) = P^n(x,t) \) is the unique solution. The result, \( P_n(x,t) = P^n(x,t) \), should come as no surprise.
(Chap 2, Ex 6c)

Clearly, the process with \( N(0) = n \) may be interpreted as the sum of \( n \) independent processes, each with \( N(0) = 1 \). That is, the state \( N(t) \), given \( N(0) = n \), equals the sum of the states \( N_t(0), N_t(1), \ldots, N_t(n) \) of independent processes with \( N(0) = \ldots = N(n) = 1 \). By a fundamental property of generating functions, then \( P_t(n, t) = P_t(0, t^n) \).

**Chapter 2, Exercise 7**

'Suppose \( S_n \) has the binomial distribution.\(^e\)

\( S_n \) is the sum of \( n \) independent Bernoulli variables, and \( S_n \) is the sum of \( n \) independent Bernoulli variables, all of which are independent and have parameter \( p \). Hence, the sum \( S_n + S_n \) is the sum of \( n + n \) independent Bernoulli variables with parameter \( p \). That is, \( S_n + S_n \) has the binomial distribution (5.1) with \( n = n + n \).

Alternatively, the generating functions of \( S_n \) and \( S_n \) are \((1 + pq^n) \) and \((1 + pq^2) \), respectively. Hence, \( S_n + S_n \) has the generating function \((1 + pq^n)(1 + pq^2) \), which is recognized as the generating function of a binomial distribution with parameters \( n = n + n \) and \( p \).

**Chapter 2, Exercise 8**

'Verify the parenthetical statement of part c of Exercise 6.'

\[
P_t(n, t) = \left[ \frac{ae^{-\lambda t}}{t!} \right]^n = e^{-\lambda t} \left( 1 - 1 - e^{-\lambda t} \right)^n
\]

\[
= e^{-\lambda t} \left( 1 + n \lambda (1 - e^{-\lambda t}) + \frac{n(n-1)}{2} \lambda^2 (1 - e^{-\lambda t})^2 + \ldots + \frac{n(n-1)(n-2)}{6} \lambda^3 (1 - e^{-\lambda t})^3 + \ldots \right)
\]

\[
= e^{-\lambda t} \sum_{k=0}^{n} \frac{n!}{k!} \lambda^k (1 - e^{-\lambda t})^k
\]

\[
= \frac{e^{-\lambda t} \sum_{k=0}^{n} \binom{n}{k} \lambda^k (1 - e^{-\lambda t})^k}{\lambda^k}
\]

Thus, \( P_t(n, t) = \binom{n}{k} e^{-\lambda t} (1 - e^{-\lambda t})^k \) for \( k \leq n \), and \( P_t(n, t) = 0 \) for \( k > n \). \( \square \)
Chapter 2, Exercise 9

'S Repeat Exercise 7, with the phrase...'

\( S_n (n=1,2) \) is the sum of \( n \) independent, identically distributed random variables following a geometric distribution. Hence, \( S_{n_1} + S_{n_2} \) is the sum of \( n_1 + n_2 \) independent, identically distributed random variables with a geometric distribution. Thus, \( S_{n_1} + S_{n_2} \) has the negative binomial distribution (58) with \( n = n_1 + n_2 \).

Alternatively, the probability generating function (p.g.f.) of \( S_n \) equals \( (p/(1-q))^n \) and the p.g.f. of \( S_{n_1} \), equals \( (p/(1-q))^{n_1} \). Hence, \( S_{n_1} + S_{n_2} \) has the p.g.f. \( (p/(1-q))^{n_1} (p/(1-q))^{n_2} = (p/(1-q))^{n_1+n_2} \), which is the p.g.f. of a variable with the negative binomial distribution (58) with \( n = n_1 + n_2 \).

Chapter 2, Exercise 10

'Feller [1971]. Find the distribution function of the length of...'

Let \( T (0 \leq T < c) \) be the length of the covering arc.

\[
F(t) = P(T \leq t) = \left( \frac{x}{c} \right)^t \quad (0 \leq t \leq c),
\]

\[
F(t) = \frac{dF(t)}{dt} = \frac{2t}{c^t} \quad (0 \leq t \leq c),
\]

\[
E(T) = \int_0^c t F(t) dt = \frac{2}{c} - e^{-2/c}.
\]

Chapter 2, Exercise 11

'Let \( X_1, \ldots, X_n \) (n \geq 2) be...'

\( R \) is the maximum of the \( n-1 \) residual variables at \( t = X_0 \). \( R \leq x \) if and only if all of these \( n-1 \) exponential variables are less than or equal to \( x \). Hence,

\[
P(R \leq x) = (1 - e^{-n x})^{n-1}.
\]
Chapter 2, Exercise 12

Let \( X_1, \ldots, X_n \) be a sequence of...

Suppose \( S_n \) is the maximum of \( n \) independent exponential variables with mean \( \mu \). Then \( P(S_n \leq t) = (1 - e^{-\mu t})^n \).

Now, \( S_n \) may be decomposed into \( n \) successive time intervals of lengths \( X_1, X_2, \ldots, X_n \), such that \( (X_i) \) is a set of independent exponential variables and \( X_i \) has mean \( \mu_i \). Since \( \sum X_i = S_n \),
\[
P(\sum X_i \leq t) = P(S_n \leq t) = (1 - e^{-\mu t})^n.
\]

Chapter 2, Exercise 13

In reliability theory, the failure rate function \( r(t) \)...

Suppose \( F(t) \) is continuous and differentiable. The probability of a failure in \( (t, t + \Delta t) \), given that no failure has occurred before \( t \), equals
\[
R(t, t + \Delta t) = \frac{F(t + \Delta t) - F(t)}{1 - F(t)},
\]
where
\[
r(t) = \lim_{\Delta t \to 0} \frac{R(t, t + \Delta t)}{\Delta t} = \frac{-R(t)}{1 - F(t)}.
\]
Thus, \( r(t) \) has the desired interpretation. If \( F(t) = 1 - e^{-\lambda t} \) \((\geq 0)\), then clearly \( r(t) = \lambda \).

Chapter 2, Exercise 4

Let \( X_1, X_2, \ldots, X_n \) be independent exponential random variables...

By an easy generalization of (5.2), \( P(\min(X_1, X_2, \ldots, X_n) \leq x) = e^{-\lambda x} \).
Thus \( Y_t = \min(X_1, X_2, \ldots, X_n) \) is exponentially distributed with parameter \( \Sigma_{i=1}^n \lambda_i \). Now write (5.2) as \( P(Y_t = \min(Y_1, Y_2)) = \frac{\mu_1}{\mu_1 + \mu_2} (1 - e^{-\mu_2 t}) \). Using the fact that \( Y_t \) and \( Y_1 \) are independent exponential variables, we find that
\[
P(Y_t = \min(Y_1, \ldots, Y_n)) = P(Y_t = \min(Y_1, Y_2)) \cdot \frac{\mu_1}{\mu_1 + \mu_2} = \frac{\mu_1}{\mu_1 + \mu_2} \cdot \frac{\mu_2}{\mu_1 + \mu_2}.
\]
Let $X_1$ and $X_2$ be independent exponential variables.

Direct proof

Clearly,

$$P\{t<X_1<z|X_1=\min(X_1,X_2)\} = e^{-\mu_1t}e^{-\mu_2z}.\]$$

Hence,

$$P\{X_1<X_2=\min(X_1,X_2)\} = \int_0^\infty e^{-\mu_1t-\mu_2z}\mu_1\mu_2 dt = \frac{\mu_1}{\mu_1+\mu_2} e^{-\mu_1t}.\]

For $t=0$,

$$P\{X_1=\min(X_1,X_2)\} = \frac{\mu_1}{\mu_1+\mu_2} \quad (5.23)$$

Thus,

$$P\{X_1>t|X_1=\min(X_1,X_2)\} = \frac{P\{X_1>t,X_2=\min(X_1,X_2)\}}{P\{X_1=\min(X_1,X_2)\}}$$

$$= e^{-\mu_1t}.$$}

Proof by use of Markov property.

The Markov property of the two exponential distributions implies

$$P(X_1=\min(X_1,X_2)|\min(X_1,X_2)>t) = P(X_1=\min(X_1,X_2)).$$

By this and the formula $P\{A|B\} = P\{A\cap B\}/P\{B\}$,

$$P\{\min(X_1,X_2)>t|\min(X_1,X_2)\} = \frac{P\{\min(X_1,X_2)>t\cap \min(X_1,X_2)\}}{P\{\min(X_1,X_2)\}}$$

which is the same as

$$P\{X_1>t|X_1=\min(X_1,X_2)\} = P\{\min(X_1,X_2)>t\}$$

Generalization to $n \geq 2$ independent exponential variables is straightforward.
Chapter 2, Exercise 16

At $t = 0$ a customer (the first customer) places a request...

(a) The derivative rate from system, and thus from line into service, equals $\mu x$ as long as any customer is in the waiting line. Hence, $X_1, X_2, \ldots, X_m$ are independent exponential variables with mean $\frac{1}{\mu}$. Then

$$E(X) = E\left(\sum_{j=1}^{m} X_j \right) = \sum_{j=1}^{m} E(X_j) = (m+1)\frac{1}{\mu}.$$

(b) $E(T) = E(X) + \frac{1}{\mu} + \frac{1}{\mu} + \cdots + \frac{1}{\mu} = m\frac{1}{\mu} + \frac{1}{\mu}.$

(c) $P\{X = m\} = \begin{cases} 0 & m = 1, 2, \ldots, s+1, \\ \frac{1}{s} & m = s+2, s+3, \ldots, m+s. \end{cases}$

(e) $P = (1 - \frac{1}{s}) \frac{1}{s}.$

Chapter 2, Exercise 17

Clearly, $P_i(t) = 1 - G_i(t)$, and $P_i(t) = \int_0^t P_i(t-s) dG_i(s)$ for $i = 1, 2, \ldots$ Assume $G(x) = 1 - e^{\lambda x}$. Then we have $P_i(t) = e^{\lambda t}$, and

$$P_i(t) = \int_0^t P_i(t-s) dG_i(s) = \int_0^t e^{\lambda (t-s)} e^{-\lambda s} ds = \lambda t e^{-\lambda t}.$$  

It is seen that (5.25) holds for $i = 0$ and $i = 1$. Suppose it holds for $i = k$, so that $P_k(t) = (k!)^{-1} \lambda^k e^{\lambda t}$. Then

$$P_{k+1}(t) = \int_0^t P_k(t-s) dG_i(s) = \int_0^t \frac{(k+1)!}{k!} \lambda^{k+1} e^{\lambda (t-s)} e^{-\lambda s} ds = \frac{k!}{k!} \lambda^k e^{\lambda t} \int_0^t e^{-\lambda y} y dy = (k!)^{-1} \lambda^k e^{\lambda t}.$$

We conclude that if $G(x) = 1 - e^{-\lambda x}$, then $P_i(t)$ is the Poisson distribution with parameter $\lambda t$. \qed
Chapter 2, Exercise 18

Prove equation (5.37) — cf. Ex. 5 of Chap. 5.

\( t \leq y \) : The event \( I_t > y \) can occur in two mutually exclusive ways:
(1) No arrivals in \([0, y)\); (2) An arrival at \( r \in (0, t]\) and no arrivals in \((r, t+y] \). Thus

\[
P(I_t \leq y) = e^{-ty} + \int_0^y e^{-ty} \, dt = e^{-ty} + yte^{-ty}.
\]

Hence,

\[
P(I_t \leq y) = 1 - e^{-ty} - yte^{-ty} \quad (t \leq y)
\]

\( t > y \) : The event \( I_t > y \) can occur in three mutually exclusive ways:
(1) No arrivals in \([0, t+y)\); (2) An arrival at \( r \in (0, t+y) \) and no arrivals in \((r, t+y] \); (3) An arrival at \( r \in (t+y, t+y) \) and no arrivals in \((r, t+y] \). Thus

\[
P(I_t > y) = e^{-ty} + \int_{t+y}^y e^{-ty} \, dt + \int_0^{t+y} e^{-ty} \, dt
\]

\[
= e^{-ty} + [e^{-ty} - e^{-ty}] + yye^{-ty}
\]

\[
= e^{-ty} + yye^{-ty}
\]

Hence,

\[
P(I_t > y) = 1 - e^{-ty} - yye^{-ty} \quad (t > y)
\]

The two equations may be combined into

\[
P(I_t \leq y) = 1 - e^{-ty} - \lambda \min(y,t)e^{-ty} \quad (5.37)
\]

Chapter 2, Exercise 19

'Let \( F(x, y) \) be the limiting joint distribution function...'

The formula

\[
F(x, y) = \lim_{\Delta x \to 0} P(R \leq x, I_t \leq y) = 1 - e^{-\lambda x} - \lambda xe^{-\lambda y} \quad (0 \leq x \leq y)
\]

may be derived from eq. (7.20) of Chapter 5.
\[ \lim_{t \to \infty} P(I, y) = \lim_{t \to \infty} P(R, y, I, y) = 1 - e^{-\lambda y} - \lambda y e^{-\lambda y}. \]

\[ \lim_{t \to \infty} P(R > x) = \lim_{t \to \infty} (1 - P(R, x)) = 1 - \lim_{t \to \infty} P(R, x) = 1 - \lim_{t \to \infty} P(R, x, I, \infty) = e^{-\lambda x}. \]

It may be shown that
\[ \lim_{t \to \infty} P(R > x, A_t > y) = \int_{x}^{\infty} \int_{y}^{\infty} f(x, y) dx dy \]
where \( f(x, y) = dF(x, y)/dx dy \) is the density function. By differentiation we find \( f(x, y) = \lambda^2 e^{-\lambda y} \). Hence,
\[ \lim_{t \to \infty} P(R > x, A_t > y) = \int_{x}^{\infty} \int_{y}^{\infty} \lambda^2 e^{-2\lambda y} dx dy \]
\[ = \left( \int_{x}^{\infty} \lambda e^{-\lambda y} dy \right) \int_{x}^{\infty} \lambda e^{-\lambda y} dy \]
\[ = e^{-\lambda x} e^{-\lambda y}. \]

\[ \lim_{t \to \infty} P(A_t > y) = \lim_{t \to \infty} P(R > 0, A_t > y) = e^{-\lambda y} e^{-\lambda y} = e^{-2\lambda y} \quad (5.34) \]
\[ \lim_{t \to \infty} P(R > x, A_t > y) = e^{-\lambda x} e^{-\lambda y} \quad \text{by (5.33)} \]
\[ = \lim_{t \to \infty} P(R > x) \lim_{t \to \infty} P(A_t > y) \quad \text{by (5.34)} \]
\[ = \lim_{t \to \infty} (P(R > x) P(A_t > y)). \]

\( f(x, y) = \lambda^2 e^{-\lambda y} \) is constant on the interval \( 0 < x < y \) for any given \( y \) and \( R_t \) is uniformly distributed throughout this covering interval.
Chapter 2, Exercise 20.

"A bus shuttles back and forth..."

By Eq. (1.5), the probability that an arbitrary passenger is one of a bus load of \( j \) people equals \( P(Y=j) = \frac{a^j}{j!} \), where \( a = \sum \frac{a^j}{j!} \). Evidently, \( P(Y=j) \equiv P(Y=j) \). It follows that, for any distribution \( \{P_j\} \),

\[
\Pi_j = \frac{\Pi_{j+1}}{a} \quad (j = 0, 1, 2, \ldots).
\]

The condition \( P_j = \Pi_j \) for all \( j \) is therefore equivalent to

\[
\begin{align*}
P_0 &= \Pi_0 = \frac{a}{a} P_1, \\
P_1 &= \Pi_1 = \frac{a}{a} P_2, \\
&\vdots \\
P_j &= \Pi_j = \frac{a}{a} P_{j+1}
\end{align*}
\]

or,

\[
\begin{align*}
P_0 &= \frac{a}{a} P_1, \\
P_1 &= \frac{2}{a} P_0, \\
&\vdots \\
P_j &= \frac{a}{a} P_{j+1}
\end{align*}
\]

As \( \sum_{j=0}^{\infty} P_j = 1 \), we must have \( P_0 = \left( \sum_{j=0}^{\infty} \frac{a^j}{j!} \right)^{-1} = e^{-a} \). The inference is

\[
P_j = \Pi_{j+1} (j = 0, 1, 2, \ldots) \Leftrightarrow P_j = \frac{a^j}{j!} e^{-a} (j = 0, 1, 2, \ldots).
\]

Chapter 2, Exercise 21.

"Suppose customers arrive according to a Poisson process..."

2. By the theorem of total probability, for \( j = 0, 1, \ldots \),

\[
P(M = j) = \int_0^\infty P(M = j | X = t) f(t) dt = \left[ \frac{a^j}{j!} \right] \int_0^\infty e^{-a t} e^{-a t} dt.
\]
(Chap. 2, Ex. 21 a)

We need the conditional means

\[ E(M|X) = \lambda X, \]
\[ E(M^2|X) = V(M|X) + E^2(M|X) = \lambda X + (\lambda X)^2. \]

Unconditioning, we derive

\[ E(M) = E(E(M|X)) = E(\lambda X) = \lambda E(X) - \lambda \tau \]
\[ E(M^2) = E(E(M^2|X)) = E(\lambda X + (\lambda X)^2) = \lambda E(X) + \lambda^2 \tau^2, \]
\[ V(M) = E(M^2) - E^2(M) = \lambda \tau + \lambda^2 \sigma^2. \]

\[ \frac{d}{dt} \frac{P(X \in \mathfrak{S} \mid M = 1)}{P(M = 1)} = \frac{(\lambda t)^1}{1!} e^{-\lambda t} dH(t), \]
\[ \frac{d}{dt} \frac{P(X \in \mathfrak{S} \mid M = 2)}{P(M = 2)} = \frac{(\lambda t)^2}{2!} e^{-\lambda t} dH(t), \]
\[ E(X|M = 1) = \int_0^\infty t dP(X \in \mathfrak{S} \mid M = 1) \]
\[ = \frac{1}{P(M = 1)} \int_0^\infty \frac{(\lambda t)^1}{1!} e^{-\lambda t} dH(t) \]
\[ = \frac{\lambda t}{P(M = 1)} \int_0^T \frac{(\lambda t)^1}{1!} e^{-\lambda t} dH(t). \]

By part (a), then,

\[ E(X|M = 1) = \frac{\lambda t}{P(M = 1)} \frac{P(M = 1)}{P(M = 1)} (\xi + \gamma_1, \ldots). \]

\[ "\|^2 - \text{We assume that} \quad P(M = 1) = \frac{(\lambda t)^1}{1!} e^{-\lambda t} (\xi + \gamma_1, \ldots). \]

By part (b),

\[ E(X|M = 1) = \frac{\lambda t}{P(M = 1)} \frac{P(M = 1)}{P(M = 1)} \]
\[ = \frac{\lambda t}{P(M = 1)} \frac{P(M = 1)}{P(M = 1)} e^{-\lambda t} \left[ \frac{\lambda t}{P(M = 1)} e^{-\lambda t} \right] \]
\[ = \tau = E(X). \]
"Only if" - We assume that \( E(X|M=\bar{1}) = \mathbb{E}(X) = \tau \) \((\bar{i} \neq 0, \bar{1}, \ldots)\)

By part (b),
\[
\tau = \frac{1}{\lambda} \sum_{i=1}^{\infty} \frac{1}{\lambda} \sum_{i=1}^{\infty} \frac{1}{\lambda} \sum_{i=1}^{\infty} \ldots = \frac{1}{\lambda} \sum_{i=1}^{\infty} \frac{1}{\lambda} \sum_{i=1}^{\infty} \ldots
\]

Hence, \( P(M=\bar{1}) = \frac{\lambda}{\lambda + \tau} P(M=\bar{1}-1) + \frac{\lambda}{\lambda + \tau} P(M=\bar{0}) \) for \( \bar{i} \neq 1 \). Utilizing \( \sum_{i=1}^{\infty} P(M=i)-1 \) we find \( P(M=\bar{0}) = e^{-\lambda \tau} \). Thus,
\[
P(M=\bar{1}) = \frac{\lambda}{\lambda + \tau} e^{-\lambda \tau} \quad (\bar{j} = 0,1,\ldots).
\]

Let \( t \) meet the condition \( 0 < H(t) < 1 \). Then
\[
P(X(t) \leq M+\bar{1}) = \frac{P(X(t) \leq M+\bar{1})}{P(M=\bar{1})} = \frac{\int_{0}^{\infty} e^{-\lambda x} dH(x)}{\int_{0}^{\infty} e^{-\lambda x} dH(x)} = \frac{\int_{0}^{t} x^{i} e^{-\lambda x} dH(x)}{\int_{0}^{\infty} x^{i} e^{-\lambda x} dH(x)}.
\]

Now define
\[
A_{\bar{i}} = \int_{0}^{t} x^{i} e^{-\lambda x} dH(x); \quad B_{\bar{i}} = \int_{0}^{t} x^{i} e^{-\lambda x} dH(x).
\]

Thus,
\[
P(X(t) \leq M+\bar{1}) = \frac{A_{\bar{i}}}{A_{\bar{i}} + B_{\bar{j}}} \quad (\bar{j} = 0,1,\ldots).
\]

Clearly,
\[
A_{\bar{j}} = \int_{0}^{t} x^{i} e^{-\lambda x} dH(x) \leq \int_{0}^{t} x^{i} e^{-\lambda x} dH(x) = tA_{\bar{i}},
\]
\[
B_{\bar{j}} = \int_{0}^{t} x^{i} e^{-\lambda x} dH(x) > \int_{0}^{t} x^{i} e^{-\lambda x} dH(x) = tB_{\bar{j}}.
\]

Hence, \( A_{\bar{j}} / A_{\bar{i}} \leq t < B_{\bar{j}} / B_{\bar{i}} \) whereby \( A_{\bar{j}} / A_{\bar{i}} < B_{\bar{j}} / B_{\bar{i}} \). Thus,
\[
B_{\bar{j}} > (A_{\bar{j}} / A_{\bar{i}}) B_{\bar{i}}, \quad \text{so that}
\]
\[
P(X(t) \leq M+\bar{1}) = \frac{A_{\bar{j}}}{A_{\bar{j}} + B_{\bar{j}}} < \frac{A_{\bar{i}}}{A_{\bar{i}} + B_{\bar{i}}}, \quad \text{and}
\]
\[
P(X(t) \leq M+\bar{1}) = P(X(t) \leq M+\bar{1}) \quad (\bar{j} = 0,1,\ldots).
\]

Thus proves the following: for all \( t \) such that \( 0 < H(t) < 1 \),
\[
P(X(t) \leq M+\bar{1}) < P(X(t) \leq M+\bar{1}) \quad (\bar{j} = 0,1,\ldots).
\]

The variable \( X \), given \( M=\bar{1}+\bar{i} \), stochastically dominates \( X \), given \( M=\bar{1} \).
It remains to be shown that the stochastic dominance implies a higher mean. A simple proof is this:

\[ E(X|M=j) = \int_0^\infty [1 - P(X \leq t | M=j)] \, dt \]
\[ > \int_0^\infty [1 - P(X \leq t | M=j)] \, dt \quad \text{(by o)} \]
\[ = E(X|M=j) \quad (j = 0, 1, \ldots). \]

We have used the fact that the mean of a non-negative variable \( X \) with distribution function \( F(t) = P(X \leq t) \) is given by the formula
\[ E(X) = \int_0^\infty [1 - F(t)] \, dt, \]
which may be proved by integration by parts.

\[ H(t) = 1 - e^{-\mu t} \] Given exponential "service time," the process may be viewed as an exponential race, repeated until the \( \mu \)-variable wins. By (5.23), the winning probabilities are \( \lambda/(\lambda+\mu) \) and \( \mu/(\lambda+\mu) \), respectively, for the \( \lambda \)- and \( \mu \)-variable. We deduce that in this case.
\[ P(M=j) = \left( \frac{\lambda}{\lambda+\mu} \right)^j \frac{\mu}{\lambda+\mu} \quad (j = 0, 1, \ldots). \]

\[ H(t) = 1 - e^{-\mu t} \] Inserting the above expression for \( P(M=j) \) into the formula in part (b) we easily derive \( E(X|M=j) = (j+1)/(\lambda+\mu) \). Alternatively, if \( M=j \), then the "service time" \( X \) is composed of \( j+1 \) intervals resulting from exponential races. By Ex. 15 these intervals are independent exponential variables with mean \((\lambda+\mu)\). Proved!

---

Chapter 2, Exercise 22

'Customers request service from a group of n servers...'

\[ a \quad P = \frac{1}{\lambda+\mu} \]
\[ b \quad P(N=j) = \left( \frac{\lambda}{\lambda+\mu} \right)^j \frac{\mu}{\lambda+\mu} \]
\[ c \quad P = \frac{\nu}{\lambda+\mu} \frac{(\mu/\lambda)^j}{\lambda+\mu (\lambda+\nu)N} \]
In Exercise 5 it was shown that

\[ g(x+y) = g(x)g(y) \] (1)

Using (1) we shall prove that

**Procedures (a) and (b) are equivalent if and only if**

\[ N_y (x=1/2) \text{ has a Poisson distribution} \]

**Proof**: We assume that \( N_y (x=1/2) \) has a Poisson distribution with mean \( \lambda \).

By procedure (a), the contents of the cells will be \( J \cdot N \) and \( K \cdot N \) respectively, and so \( J \) and \( K \) are independent Poisson variables with means \( \lambda \). By procedure (b), \( N \cdot (J+K) \) is a Poisson variable with mean \( 2\lambda \). The decomposition property expressed by (1), implies that \( J \) and \( K \) will be independent Poisson variables with means \( \lambda + \lambda = 2\lambda \). Hence, procedures (a) and (b) are equivalent.

**Only if**. We assume that procedures (a) and (b) are equivalent.

By (1),

\[ g(x)g(y) = g(x+y) \] (2)

Hence,

\[ g(x+y) = g(x)g(y) \] (3)

This implies \( g(0) > 0 \). Now put

\[ g(0) = u(0)g(0) \] (4)

Inserting (3) into (2) yields

\[ u(x)u(y) = u(x+y) \] (5)

Since \( g(x) \) is increasing in \( x \), \( u(x) \) is \( u(x) \), by (1). Eq. (4) is identical to Eq. (5). It follows that the only increasing function \( u \) satisfying the functional equation (4) is of the form

\[ u(x) = e^{x\theta} \quad (\theta > 0) \].

Chapter 2, Exercise 23
Thus, by (5),
\[ g(\alpha) = e^{\alpha^2} g(0). \]  
(5)

But \( g(1) = 1 \), so by (5) \( 1 = e^{\alpha} g(0) \), whence \( g(0) = e^{-\alpha} \). Thus
\[ g(\alpha) = e^{-\alpha(\alpha-1)} \quad (\alpha > 0). \]  
(6)

This is recognized as the p.d.f. of a Poisson variable with mean \( \alpha \). Thus \( N_j (j=1,2) \) has a Poisson distribution.

**Chapter 2, Exercise 24**

'Consider the single-server queue with an unlimited number.'

\[ \textbf{A} \] First multiply equation (1) \((i=0,1,...)\) of equation system (1) by \( \alpha^i \):
\[
\begin{align*}
\Pi^*_0 \alpha^i &= \alpha^i \Pi^*_0 + \sum_{i=1}^{\infty} (\rho_i \Pi^*_1)^i \\
\Pi^*_1 \alpha^i &= \alpha^i \Pi^*_0 + \sum_{i=1}^{\infty} (\rho_i \Pi^*_2 + \rho_i \Pi^*_1)^i \\
\Pi^*_2 \alpha^i &= \alpha^i \Pi^*_0 + \sum_{i=1}^{\infty} (\rho_i \Pi^*_3 + \rho_i \Pi^*_2 + \rho_i \Pi^*_1)^i
\end{align*}
\]

Adding all these equations results in
\[ y(\alpha) = h(\alpha) \Pi^*_0 + \alpha^{-1}(\rho_0 + \rho_1 \alpha + \rho_2 \alpha^2 + ...) \sum_{i=1}^{\infty} \Pi^*_i \alpha^i = h(\alpha) \Pi^*_0 + \alpha^{-1} h(\alpha) [g(\alpha) - \Pi^*_0] \]

Hence,
\[ g(\alpha) = \frac{(\alpha-1) h(\alpha) \Pi^*_0}{\alpha - h(\alpha)} \]  
(5)

\[ \textbf{B} \]
\[ h(\alpha) = \sum_{i=0}^{\infty} \rho_i \Pi^*_i = \sum_{i=0}^{\infty} \left( \frac{e^{-\lambda \alpha}}{\lambda \alpha} \alpha^i \right) \lambda^i e^{-\lambda H(3)} \]
\[ = \sum_{i=0}^{\infty} \left( \frac{e^{-\lambda \alpha}}{\lambda \alpha} \alpha^i \right) \lambda^i e^{-\lambda H(3)} = \left( \frac{e^{-\lambda \alpha}}{\lambda \alpha} \right) \lambda e^{-\lambda H(3)} \]
\[ = \frac{e^{-(\alpha-1) \lambda}}{\lambda \alpha} \lambda e^{-\lambda H(3)} = \gamma(\lambda-\alpha). \]
(Chap. 2, Ex. 24 b)

Substitution of \( h(z) = \eta(z, x) \) into (3) yields

\[
g(z) = \frac{(z-1)\eta(z, x)}{z - \eta^3(z, x)} \Pi^*  
\]

(4)

By the application of l'Hospital's rule to Eq. (9),

\[
g(1) = \lim_{z \to 1} g(z) = \left[ \frac{\frac{d}{dz} \left( (z-1)\eta(z, x) \right)}{\frac{d}{dz} (z - \eta^3(z, x))} \right]_{z=1} \Pi^* 
\]

\[= \frac{(z-1)\eta(z, x) + \eta^3(z, x)}{1 - 3\eta^2(z, x)} \Pi^* \]

\[= \frac{\eta(z, x)}{1 - 3\eta(z, x)} \Pi^* = \frac{\Pi^*}{1 - \Pi(z, x)}. \]

Clearly,

\[g(1) = 1.\]  

(5)

Hence,

\[\Pi^*_0 = 1 - \Pi(1).\]  

(6)

Now, \( \Pi(z, x) = \sum_1 \beta_i \) is the mean number of arrivals during a service time, which in Exercise 21 a was shown to be equal to \( \lambda \tau \). That is, \( \Pi(1) = \lambda \tau \). For \( \lambda \tau = \phi < 1 \) then,

\[\Pi^*_0 = 1 - \phi.\]  

(6)

Thus Eq. (4) becomes

\[g(z) = \frac{(z-1)\eta(z, x)}{z - \eta(z, x)} (1 - \Phi) \quad (\phi < 1).\]  

(7)

By (4,5), \( E(N^*) = g(1) \). Differentiation of (5), with \( \Pi^*_0 \) replaced by \( 1 - \phi \), gives

\[g'(z) = \frac{A(z)}{B(z)} (1 - \phi),\]

where

\[A(z) = h(z) - \eta^3(z, x) - (z-1)\eta(z, x),\]

and

\[B(z) = (z - h(z))^2.\]
Since \( h(1) = 1 \), \( A(1) = 0 \) and \( B(1) = 0 \). Thus, evaluation of \( g(0) \) requires the application of L'H魀ital's rule. Differentiation yields

\[
A'(a) = 2a h'(a) - 2h(a)h''(a) - a(1-a)h'(a),
\]

\[
B'(a) = 2(1-a)h'(a)(1-h'(a)).
\]

As \( A'(1) = 0 \) and \( B'(1) = 0 \) we differentiate once more:

\[
A''(a) = 2h'(a)(1-h'(a)) + (h'_a - 1 - 2h(a)h''(a) - a(1-a)h'(a),
\]

\[
B''(a) = 2(1-h'(a))^2 - 2(a-h'(a))h''(a).
\]

\( A''(a) \) and \( B''(a) \) have to be evaluated at \( a = 1 \). We already know that \( h(1) = 1 \) and \( h'(1) = -p \). In order to find \( h''(1) \), recall that by part (b)

\[
h_{(a)} = \eta(\lambda-\lambda_0). \quad \text{Hence, } h_{(a)} = -\lambda \eta(\lambda-\lambda_0) \quad \text{and } h_{'(a)} = \lambda^2 \eta''(\lambda-\lambda_0).
\]

Thus \( h''(1) = \lambda^2 \eta''(0) \). By definition, \( \eta(0) = \int_0^\infty e^{-\lambda_0} dH(\lambda) \). Hence,

\[
\eta''(0) = \int_0^\infty \lambda^2 e^{-\lambda_0} dH(\lambda)
\]

and \( \eta''(0) = \int_0^\infty \lambda^2 e^{-\lambda_0} dH(\lambda) \). Thus \( \eta''(0) = 2^\lambda \). By these results,

\[
A''(1) = 2p(1-p) + \lambda^2 (\sigma^2 + \tau^2),
\]

\[
B''(1) = 2(1-p)^2.
\]

By L'H魀ital's rule, \( g(0) = \frac{A''(1)}{B''(1)} (1-p) \). Hence

\[
E(N^*) = p + \frac{2p(1-p)(\sigma^2 + \tau^2)}{2(1-p)}. \quad (8)
\]

Let \( \delta = -e^{-\lambda_0} \). Then \( \eta(\lambda) = \int_0^\infty \delta e^{-\lambda_0} dH(\lambda) = \int_0^\infty e^{-\lambda_0} d\mu \mu / (\mu + \omega) \). Thus \( \eta(\lambda-\lambda_0) = \mu / (\mu + \lambda - \lambda_0) \) and, by (7),

\[
g(\sigma) = \frac{(1-p) \mu (\sigma + \lambda_0)}{\delta - \mu / (\mu + \lambda - \lambda_0)} (1-p) = \frac{(1-p) \mu}{(\delta - \mu / (\mu + \lambda - \lambda_0)} (1-p)
\]

\[
= (1-p)/(1-p_2) = \sum_{j=0}^\infty (1-p)\rho^j j!
\]

Hence,

\[
T^*_j = (1-p)^j j! \quad (j = 0, 1, \ldots). \quad (9)
\]
Chapter 2, Exercise 25

"An operations research consultant..."
No comment.

Chapter 2, Exercise 26

"In the model of Exercise 25, let $X$ be the merging time..."

In the model of Exercise 25 the (a priori) interarrival times $U_1, U_2, \ldots$ are i.i.d. exponential variables with mean $\lambda^{-1}$, and the required acceleration times $V_1, V_2, \ldots$ are i.i.d. exponential variables with mean $\rho^{-1}$. As a consequence, the merging time $X = U_1 + U_2 + \ldots + U_n + V_n$ is exponentially distributed with mean $\lambda^{-1}$ and does not depend on $\alpha$ (see preceding pages of book).

In the model of this exercise, it will be assumed that $V_1 = V_2 = \cdots = V_k$ where $V$ is an exponential variable with mean $\rho^{-1}$. In this case, the mean and variance of $X$ are a function of $\alpha$. Both mean and variance are greater than in the model of Exercise 25 for identical $\alpha$ and $\rho$.

2) Let $V$ equal the constant $c$. Then

$$X = c + \frac{c}{\lambda} U,$$

where each of the observed $U$'s $(i=1, \ldots, n)$ has the conditional distribution of $U$ given $U \leq c$. Since these $U$'s are i.i.d. variables and independent of $n-1$,

$$E(X|V=c) = c + E(n-1)E(U|U \leq c).$$

Now, $n-1$ has the geometric distribution with parameter $p = \Pr(U>c) = e^{-\lambda c}$. Hence $E(n-1) = (1-p)/p = e^{\lambda c} - 1$. Substitution into (2) yields the desired expression

$$E(X|V=c) = c + (e^{\lambda c} - 1)E(U|U \leq c).$$
(Chap. 2, Ex. 26 b)

1. First we determine $E(U|U ≤ c)$. By assumption, $U$ is exponentially distributed with mean $\alpha$; i.e. $f_U(u) = \frac{1}{\alpha} e^{-\frac{u}{\alpha}}$.

Hence, $E(U|U ≤ c) = \frac{\int_0^c u f_U(u) du}{\Pr(U ≤ c)} = \frac{\frac{c}{\alpha} e^{-\frac{c}{\alpha}}}{1 - e^{-\frac{c}{\alpha}}}$, and the density function of $U$, given $U ≤ c$, is $f_U(u|U ≤ c) = \frac{u}{\alpha} e^{-\frac{u}{\alpha}}(1 - e^{-\frac{c}{\alpha}})$ for $0 ≤ u ≤ c$.

Consequently,

$$E(U|U ≤ c) = \frac{\int_0^c u f_U(u) du}{\int_0^c f_U(u) du} = \frac{\frac{c}{\alpha} e^{-\frac{c}{\alpha}}}{1 - e^{-\frac{c}{\alpha}}} = \frac{1}{\alpha} - \frac{c}{\alpha e^{\frac{c}{\alpha}}}. \quad (4)$$

Hence,

$$E(U|U ≤ c) = \frac{1}{\alpha} - \frac{c}{\alpha e^{\frac{c}{\alpha}}}. \quad (5)$$

Substitution of Equation (4) into Equation (5) results in

$$E(X|V=c) = \frac{e^{\frac{c}{\alpha}} - 1}{\alpha}. \quad (6)$$

We now assume that $V$ is exponentially distributed with mean $\beta$. Unconditioning on $V$ we find

$$E(X) = \int_0^\infty E(X|V=v) f_V(v) dv = \frac{\beta}{\alpha} \left( \int_0^\infty e^{-\frac{v}{\alpha}} dv - \int_0^\infty e^{-\frac{v}{\beta}} dv \right),$$

by which

$$E(X) = \begin{cases} (\beta-\alpha)^{-1}, & \text{when } \beta > \alpha, \\ \infty, & \text{when } \beta \leq \alpha. \end{cases} \quad (7)$$

2. For the purpose of calculating $V(X)$ we shall employ the decomposition formula

$$V(X) = E(V(X)|V=c) + V(E(V(X)|V=c)), \quad (8)$$

which gives $V(X)$ as the sum of the mean of the conditional variance, given $V$, and the variance of the conditional mean, given $V$. Clearly,

$$V(X|V=c) = V(c + \sum_{i=1}^k U_i) = V(\sum_{i=1}^k U_i).$$

The distribution of $\sum_{i=1}^k U_i$ is a *compound* distribution. Since the formulas in part (b) of Exercise 4 also hold for non-discrete variables (see Chap. 5, Ex. 5) we have
(Chap. 2, Ex. 26-6)

\[ V(X|V=c) = E(U|V=c) - V(U) - E^2(U|V=c) \]  

(8)

By part (a), \( n-1 \) is geometrically distributed with parameter \( p = e^{-c} \)

Hence, \( E(n-1) = e^{-c} - 1 \) and \( V(n-1) = (1-p)/p = e^{c(e^{-c}-1)}. \) \( E(U|V=c) \)

is given by Eq. (9). Only \( V(U|V=c) \) remains to be calculated.

Following the development in part (b) we find

\[ V(U|V=c) = E(U^2|U\leq c) - E^2(U|U\leq c) \]

\[ = \left\{ \begin{array}{l}
\int_{0}^{c} c^2 U^2 dU - \left( \frac{1}{\alpha} - \frac{c}{\alpha^{c-1}} \right)^2 \\
\frac{1}{\alpha^2} \left( \frac{ax^2 + 2\alpha x + 2\alpha}{\alpha^2(1-e^{-\alpha x})} \right) \left( \frac{1}{\alpha} - \frac{c}{\alpha^{c-1}} \right)^2.
\end{array} \right. \]

Hence,

\[ V(U|U\leq c) = \frac{1}{\alpha^2} - \frac{c^{c-1} e^{-c}}{(e^{c-1})^2}. \]  

(9)

Substitution of the various expressions into (8) gives

\[ V(X|V=c) = (e^{-c})(\frac{1}{\alpha} - \frac{c^{c-1} e^{-c}}{(e^{c-1})^2}) + e^{c(e^{-c}-1)} \left( \frac{1}{\alpha} - \frac{c}{\alpha^{c-1}} \right)^2, \]

that reduces to

\[ V(X|V=c) = \frac{1}{\alpha} \left( e^{2ac} - 2ace^{ac} - 1 \right). \]  

(10)

Assuming that \( V \) is exponentially distributed with mean \( \beta^{-1} \)

we have

\[ E_{\rho}(V(X|V=c)) = \int_{0}^{\rho} V(X|V=c) e^{-\lambda c} dc \]

\[ = \left. \int_{0}^{\rho} \frac{e^{2ac} - 2ace^{ac} - 1}{e^{\alpha x}} e^{-\lambda c} dc \right|_{\rho}^{0} \]

\[ = \frac{\lambda}{\alpha^2} \left( \int_{0}^{\rho} e^{-\alpha x} dx - 2\alpha e^{-\alpha x} \int_{0}^{\rho} e^{-\lambda c} dc - 2\alpha \int_{0}^{\rho} e^{-\lambda c} dc - \int_{0}^{\rho} e^{-\lambda c} dc \right) \]

If \( \rho \leq 2\alpha \), then \( E_{\rho}(V(X|V=c)) = \infty. \) If, on the other hand, \( \rho > 2\alpha \), then

\[ E_{\rho}(V(X|V=c)) = \frac{\alpha}{\alpha} \left( \frac{1}{\rho - 2\alpha} - 2\alpha \left( \frac{1 - e^{-\lambda \rho}}{\rho - 2\alpha} \right) \right) \]

\[ = \frac{\alpha}{\rho - 2\alpha} - \frac{2\alpha}{(\rho - 2\alpha)} - \frac{1}{\rho}. \]
Hence,\[
E_c(V(X)|V=c) = \begin{cases} 
\frac{2\alpha}{(\alpha-2\lambda)(\alpha-3\lambda)} & \text{when } \alpha > 2\lambda, \\
\infty & \text{when } \alpha \leq 2\lambda.
\end{cases} \tag{11}
\]

Still under the assumption of an exponentially distributed \( V \) with mean \( \alpha^{-1} \), we derive the second term on the right-hand side of (7):\
\[
V_c(E(X)|V=c) = \int_0^\infty (E(X|V=c)-E(X))^2 \rho e^{-\rho v} dv
\]

By (5), (6)\
\[
= \frac{2}{\alpha^2} \int_0^\infty \left( \frac{e^{-\alpha v} - \frac{1}{\rho} - \frac{1}{\rho} - \frac{2}{\alpha} \frac{\rho}{\alpha-\lambda} (\frac{1}{\rho} - \frac{1}{\rho}) + \frac{1}{(\alpha-\lambda)^2}}{\alpha} \right) e^{-\rho v} dv
\]

If \( \rho \leq 2\lambda \), then \( V_c(E(X|V=c)) = \infty \). Otherwise
\[
V_c(E(X|V=c)) = \frac{2}{\alpha^2} \left( \frac{\rho}{\alpha - 2\lambda} - \frac{2}{\alpha} + \frac{1}{\rho} - \frac{2\alpha}{\alpha(\alpha-\lambda)} (\frac{1}{\rho} - \frac{1}{\rho}) + \frac{1}{(\alpha-\lambda)^2} \right).
\]

Hence,
\[
V_c(E(X|V=c)) = \begin{cases} 
\frac{\rho}{\alpha^2} & \text{when } \rho > 2\lambda, \\
\infty & \text{when } \rho \leq 2\lambda. \tag{12}
\end{cases}
\]

By adding Equations (11) and (12) according to the decomposition formula, Eq. 7, we finally obtain, for the case of exponentially distributed characteristic values:
\[
V(X) = \begin{cases} \frac{\rho + 2\alpha}{\rho - 2\alpha} \frac{1}{(\alpha-\lambda)^2} & \text{when } \rho > 2\lambda, \\
\infty & \text{when } \rho \leq 2\lambda. \tag{13}
\end{cases}
\]
A single-server queueing system.

By Equation (1.1),
\[ P_0 = \left[ 1 + \sum_{j=1}^{\infty} \frac{\lambda_j \lambda_{j+1} \cdots \lambda_m}{m_j m_{j+1} \cdots m_m} \right]^{-1} = \left[ 1 + \sum_{j=1}^{\infty} \frac{\lambda_j (\mu_0)^j}{j!} \right]^{-1} = e^{-\lambda_0 \mu_0}, \]
and for \( j = 1, 2, \ldots, \)
\[ P_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{m_0 m_1 \cdots m_{j-1}} P_0 = \frac{(\lambda_0)^j}{j!} e^{-\lambda_0 \mu_0}. \]

Since \( \mu^{-1} \sim \tau, \)
\[ P_j = \frac{(\lambda_0)^j}{j!} e^{-\lambda_0 \mu} \quad (j = 0, 1, \ldots). \]

By Equation (2.6), for \( j = 0, 1, \ldots, \)
\[ \pi_j = \lambda_j P_j \sum_{k=0}^{\infty} \pi_k = \frac{\lambda_j (\lambda_0)^j}{j!} e^{-\lambda_0 \mu} + \sum_{k=0}^{\infty} \frac{\lambda_0^k}{k!} e^{-\lambda_0 \mu} = \frac{\sum_{k=0}^{\infty} \lambda_0^k}{\sum_{k=0}^{\infty} \lambda^k} e^{-\lambda_0 \mu} e^{\lambda_0 \mu} = \frac{\lambda_j \rho_j}{\pi_j}. \]

Hence,
\[ \pi_j = (1 - e^{-\lambda_0 \mu})^{-1} \pi_0 \quad \text{for} \quad j = 0, 1, \ldots. \]

As \( s \to 1, \) by (19) the carried load is
\[ a^* = \sum_{j=1}^{\infty} P_j = 1 - P_0 = 1 - e^{-\lambda_0 \mu}. \]

Obviously, the mean arrival rate is \( \lambda = \sum_{j=1}^{\infty} \lambda_j \pi_j, \) so
\[ \lambda = \sum_{j=1}^{\infty} \frac{\lambda_0^j}{j!} e^{-\lambda_0 \mu} = \tau (1 - e^{-\lambda_0 \mu}). \]

It follows that the offered load is \( \lambda = \pi \tau \times 1 - e^{-\lambda_0 \mu}, \) and \( a = a^*. \)

Here the arrival rate is \( \lambda, \) but the effective arrival rate - concerning arrivals affecting a change of state - in state \( j \) is \( \lambda_j = \lambda \left( 1 - P_j \right) / \left( 1 - \pi_j \right) = \lambda / \pi_j. \) Also, \( m_j = \mu_j \). Thus, the queueing system can be modeled as a birth-and-death process with the same parameters as the model of part (a). Consequently, \( (P_j) \) is as in part (a). Furthermore, since the arrival process is Poisson, \( \pi_j = P_j. \) We conclude that
(Chap. 3, Ex. 1 b)

\[ P_1 = \frac{\mu}{\mu + \lambda} \quad (\lambda = \alpha, \alpha') \]

As in part (a),

\[ \alpha' = 1 - e^{-\lambda \tau} \]

However, the offered load is

\[ \alpha = \alpha \tau \]

Letting \( P \) denote the probability that an arbitrary arrival does not receive service,

\[ P = \frac{\alpha}{\lambda + \alpha} P_1 = \frac{\alpha}{\lambda + \alpha} \frac{1 - e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} \]

\[ = 1 - \frac{e^{-\alpha \tau}}{\alpha} = 1 - \frac{\alpha'}{\alpha} = \frac{\alpha - \alpha'}{\alpha} \]

(c) As in the models of parts (a) and (b), \( \lambda / \mu_1 = \lambda / (\mu_1 + \mu_2) \), so the state distribution \( P_1 \) is the same in these cases. Furthermore, since the arrival process is Poisson, \( P_1 = P_1 \).

Hence,

\[ P_1 = P_1 = \frac{\alpha}{1 + \alpha} e^{-\lambda \tau} \quad (\lambda = \alpha, \alpha') \]

Also, as in parts (a) and (b), the carried load is

\[ \alpha' = 1 - e^{-\lambda \tau} \]

The birth-and-death process will not be affected by preemption coupled with service in reverse order of arrival. In this case, a customer who arrives at state \( j \) will be served at rate \( \mu_1 + \mu_2 \), where in service, and his mean service time, which is not affected by preemption, is \( 1 / (\mu_1 + \mu_2) \). Then the arrival process service times equals

\[ \lambda = \frac{\lambda}{\lambda + \mu_1} \mu_1 + \frac{\lambda}{\lambda + \mu_2} \mu_2 + \frac{1}{\lambda + \mu_1 + \mu_2} \mu_1 + \frac{1}{\lambda + \mu_1 + \mu_2} \mu_2 e^{-\lambda \tau} \]

The offered load is therefore \( \alpha' = \alpha \tau - \frac{1}{\lambda} (1 - e^{-\lambda \tau}) \). Hence,

\[ \alpha = \alpha' \tau \]
Chapter 3, Exercise 2

"Customers arrive at a two-chair shoe-shine stand..."

(a) \( \lambda = 10, \mu = 10, s = 1, k = 1 \)

The corresponding birth-and-death model has: \( \lambda_0 = \lambda_1 = \lambda = 10, \)
\( \lambda_2 = 0; M_0 = 0, M_1 = \mu = M = 10. \) By (11),
\[ P_0 = 1 + \frac{2a}{M_1 + \lambda_1 k} \]
\[ P_1 = \frac{2a}{M_1} P_0, \]
\[ P_2 = \frac{2a}{M_1 M_2} P_0. \]

Hence,
\[ (P_0, P_1, P_2) = (\frac{1}{3}, \frac{5}{3}, \frac{4}{3}). \]

(b) The mean number of customers served per hour is
\[ \Delta = M_1 P_1 + M_2 P_2 = \frac{10}{3} + \frac{8}{3} = 6.67 \]

(c) \( \lambda = 10, \mu = 10, s = 2, k = 0 \)

The corresponding birth-and-death model has: \( \lambda_0 = \lambda_1 = \lambda = 10, \)
\( \lambda_2 = 0; M_0 = 0, M_1 = \mu = M = 10. \) Applying the above formulas
we find
\[ (P_0, P_1, P_2) = (\frac{1}{10}, \frac{9}{10}, \frac{1}{10}) \]

and
\[ \Delta = 8 \frac{1}{10}. \]

Chapter 3, Exercise 3

"Derive (3.5) from the definition (14) and the probabilities (3.3)"

\[ a' = \sum_{i=1}^{s} P_i = \sum_{k=0}^{s} a^{\lambda k} / k! = a \frac{\sum_{k=0}^{s} a^{\lambda k} / k!}{\sum_{k=0}^{s} a^{\lambda k} / k!} \]

\[ = a \frac{\sum_{k=0}^{s-1} a^{\lambda k} / k!}{\sum_{k=0}^{s} a^{\lambda k} / k!} = \frac{a}{1 - \frac{a^{\lambda / \mu}}{\sum_{k=0}^{s} a^{\lambda k} / k!}} \]

By (3.4), then
\[ a' = a [1 - B(s, a)] \quad (3.5) \]
Chapter 3, Exercise 4

'Consider an Erlang loss system with 10 servers.'

The solution requires the evaluation of $B(a, a)$. Figures A-1 and A-2 of Appendix A provide the answers. Alternatively, one can use a table of the cumulative Poisson distribution, since

$$B(a, a) = \frac{a^b e^{-a}}{\sum_{k=0}^{\infty} \frac{a^k}{k!} e^{-a}},$$

and both numerator and denominator may be read off or easily calculated from a table with cumulative Poisson probabilities. We find

$$B(10, 9.5) = \frac{0.0049}{0.9951} = 0.00495,$$

and $B(10, 9) = 0.0058$. Accordingly, we accept $a = 4.5$ as an approximate solution of $B(10, a) = 0.01$. We also find

$$B(16, 9.0) = 0.0100, \quad B(17, 9.0) = 0.0058.$$

Thus, a doubling of the offered load does not necessitate a doubling of the number of servers, from 10 to 20, in order to prevent service degradation. Only 7 servers need be added to the system.

Chapter 3, Exercise 5

'An entrepreneur offers services...'

The offered load is $a = \lambda R = 4 \times 4 = 16$. The hourly profit of operating cost $c$ equals $H(s, c) = \lambda (1 - B(s, a)) e^{-(s - c)} = 10 (1 - B(s, 16)) e^{-s}$. Hence,

<table>
<thead>
<tr>
<th>$s$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(s, 16)$</td>
<td>0.89</td>
<td>0.65</td>
<td>0.45</td>
<td>0.31</td>
<td>0.19</td>
<td>0.117</td>
<td>0.065</td>
<td>0.038</td>
<td>0.015</td>
<td>0.005</td>
</tr>
<tr>
<td>$H(s, 16)$</td>
<td>0.10</td>
<td>0.15</td>
<td>0.24</td>
<td>0.34</td>
<td>0.41</td>
<td>0.48</td>
<td>0.56</td>
<td>0.63</td>
<td>0.68</td>
<td>0.76</td>
</tr>
</tbody>
</table>

Thus, at $c = 10$, the optimal number of servers is 5, and the corresponding profit rate equals 3.0. The break-even point for $c = 10$ is $s = 11$. If the entrepreneur will just break even, with $s > 11$, he will lose.

□
Chapter 3, Exercise 6

'Show that \( B(s, a) = \frac{aB(s-1, a) + s + aB(s-1, a)}{a + sB(s-1, a)} \)'

By (3.4), for \( a \geq 1 \),

\[
B(s, a) = \frac{a^s / a!}{\sum_{k=0}^{s} a^k k^s} = \frac{a^s / a!}{\sum_{k=0}^{s} k^s a^k / k!} \\
- \frac{a}{s} B(s-1, a) [1 - B(s, a)],
\]

where \( B(0, a) = 1 \). Solving for \( B(s, a) \) we derive

\[
B(s, a) = \frac{aB(s-1, a)}{s + aB(s-1, a)} \quad (s = 1, 2, \ldots).
\]

Chapter 3, Exercise 7

'Consider an Erlang loss system with retrials.'

No comment.

Chapter 3, Exercise 8

'Consider an equilibrium \( s \)-server Erlang loss system...'

In the Erlang loss system the event (next arrival is blocked) will occur if and only if (a) the server is busy, and (b) next arrival occurs before next service completion. Obviously, event (a) has probability \( B(s, a) (1 - p_a - r_b) \).

Given (a), event (b) has probability \( \lambda / (\lambda + \mu) \), by Eq. (8.23) of Chapter 2, as time to next arrival and time to next service completion are independent exponential variables with parameters \( \lambda \) and \( \mu \), respectively. Hence

\[
p = B(s, a) \frac{\lambda}{\lambda + \mu} = \frac{a}{a + s} B(s, a).
\]

The reason \( p \) is not equal to \( B(s, a) \), as one might naively think, is that "next arrival" is not an arbitrary arrival.
Chapter 3, Exercise 9

We consider the $s$-server Erlang loss system with exponential service times, and let $\lambda = \text{arrival rate}$ and $\mu = \text{service rate}.$

(a) Clearly,

$$P_i = \begin{cases} 
0 & (i = j \neq 1), \\
\frac{\lambda^i}{\lambda + i \mu} & (0 \leq i \leq s-1, j = i+1), \\
\frac{1}{\lambda + i \mu} & (1 \leq i \leq s-1, j = i-1), \\
\frac{1}{\lambda + i \mu} & (i = s, j = s-1). 
\end{cases}$$

(b) Clearly,

$$m_i = \begin{cases} 
(\frac{1}{\lambda + i \mu}) & (0 \leq i \leq s-1), \\
\frac{1}{\lambda \mu} & (i = s). 
\end{cases}$$

(c) Substituting Eq. (6) into Eq. (3) yields

$$P_0^* = \frac{\mu}{\lambda + \mu \mu} P_1^*$$

$$P_i^* = \frac{\lambda}{\lambda + (i-1) \mu} P_{i-1}^* + \frac{(i-1) \mu}{\lambda + (i-1) \mu} P_i^* \quad (i \geq 2, 1 \leq i \leq s-2),$$

$$P_{i-1}^* = \frac{\lambda}{\lambda + (i-1) \mu} P_i^* \quad (i > 1),$$

$$P_{i-1}^* = \frac{\lambda}{\lambda + (i-1) \mu} P_i^* \quad (i > 1).$$

By recursive solution we obtain

$$P_i^* = \begin{cases} 
\frac{\lambda^i \mu^i}{\lambda} \cdot \frac{(\lambda \mu)^{s-i}}{(s-1)!} P_0^* & (0 \leq i \leq s-1), \\
\frac{1}{\lambda} \cdot \frac{(\lambda \mu)^{s-i}}{(s-1)!} P_0^* & (i = s). 
\end{cases}$$

Now, by (7) and (8),

$$m_i P_i^* = \frac{1}{\lambda} \cdot \frac{(\lambda \mu)^{s-i}}{(s-1)!} P_0^* \quad (0 \leq i \leq s).$$

Inserting this expression into Eq. (5), with $k = s,$ we derive Eq. (33):
Chapter 3, Exercise 10

'Two independent Poisson streams of traffic...'

Let the high priority stream parameters be \( \lambda_1 \) and \( \tau_1 \), where \( \lambda_1 = 80 \) and \( \tau_1 = 0.2 \), and let the low priority stream parameters be \( \lambda_2 \) and \( \tau_2 \). The two service time distributions may be general. For \( s = 10 \), the average overflow rate of high priority customers is known to be \( \lambda_1 \tau_1 = 2 \). We wish to determine \( \alpha = \frac{\lambda_1 \tau_1}{\tau_2} \).

The primary group serves two independent streams of Poisson traffic on a BCC basis. Therefore, as argued in the text, the primary system is an Erlang loss system with arrival rate \( \lambda = \lambda_1 \lambda_2 \) and a mixed service-time distribution with the mean \( \tau = \frac{\lambda_1 \tau_1 + \lambda_2 \tau_2}{\lambda_1 + \lambda_2} \). The total offered load is \( \alpha = \lambda \tau - \lambda_1 \tau_1 - \lambda_2 \tau_2 = 20 \times 0.2 + \lambda_2 \tau_2 = 4 + \alpha_2 \). The percentage overflow of high priority customers clearly is \( \lambda_1 / \lambda = 2 / 20 = 0.1 \). The same percentage will overflow from each stream arriving at the primary group, so \( B(a, a) = 0.1 \). That is, \( B(a, a) = 0.1 \).

Solving by use of the graph in Appendix A-I, we find \( a = 0.5 \) \( (B(0, 0.5) = 0.0995) \). Hence \( a_1 = a - a_2 = 0.5 - 0.1 = 0.4 \). Thus, \( \alpha = 2.5 \).

\[ \begin{align*} 
\alpha &= 2.5 \\
\lambda_1 \tau_1 &= 2 \lambda_2 \\
\therefore \alpha &= \lambda_1 \tau_1 = 2 \lambda_2 \tau_2 = 2 \lambda_2 = 0.5 \\
\text{It follows that the new overflow rate of high priority customers will be} \\
\lambda_1^* &= \lambda_1 B(a_2^*) = 20 \times B(10, 0.1) = 20 \times 0.26 = 5.2 \\
\text{The factor of increase is} \\
\frac{\lambda_1^*}{\lambda_1} &= \frac{5.2}{2} = 2.6 \\
\text{It is not permissible to design the backup group by use of Erlang's loss formula which assumes Poisson traffic. The overflow traffic is not Poisson. Disregarding this fact will lead to underestimation of the loss on the backup group, one would think.} 
\end{align*} \]
Chapter 3, Exercise 11

Prove equation (3.12).

For all $t > 0$, 
\[ 0 \leq t[1-H(t)] = \int_0^t \omega \, dH(x) \leq \int_0^\infty xdH(x). \]

Now, $\mu^{-1} = \int_0^\infty xdH(x) < \infty$ implies $\lim_{t \to \infty} \int_0^t xdH(x) = 0$. Hence, taking limits we obtain

\[ 0 \leq \lim_{t \to \infty} t[1-H(t)] \leq \lim_{t \to \infty} \int_0^t xdH(x) = 0. \]

Thus,
\[ \lim_{t \to \infty} t[1-H(t)] = 0. \quad (3.12) \]

Chapter 3, Exercise 12

1. Each customer stays in the system (queue + service) for a time $T$ that follows the sojourn time distribution $H(x)$. Thus the queuing system may be modeled as an infinite-server queue where the sojourn time is interpreted as a service time. It follows that $\{\bar{D}(t)\}$ is the Poisson distribution
\[ \bar{P}_k(t) = \frac{(\lambda t p(t))^k}{k!} e^{-\lambda t p(t)} \quad (k = 0, 1, \ldots). \quad (3.11) \]
with
\[ p(t) = 1 - H(t) + \left( \frac{t}{\mu} \right) dH(x), \quad (3.9) \]
where now $H(x)$ is the sojourn time distribution function.

2. Assume that $T$ has the exponential distribution with mean $\mu^{-1}$. Customers may defect before reaching the server. For a customer who does enter service, the remaining sojourn time (service time) will, by the Markov property, also be exponentially distributed with mean $\mu^{-1}$. 

(Chap. 3, Ex. 12 c)

As in part (b), let \( T \) follow the exponential distribution. Let \( \delta \) denote the mean detection rate (from queue). In state \( j \geq 0 \), the detection rate is \((1-\delta)\mu\). Hence

\[
\delta = \sum_{j=1}^{\infty} (j-1) \mu P_j
\]

By (3.8),

\[
P_j = \frac{\lambda(\alpha-1)}{\mu} e^{-\alpha} (j \geq 0, 1, 2, \ldots).
\]

\(\mu^{-1}\) is both mean service time and mean service time (in the normal sense), so \(\alpha = \lambda/\mu\) is the offered load. Thus

\[
q = \frac{\lambda}{\mu} - \frac{1}{\mu} \sum_{j=0}^{\infty} (j-1) \mu \frac{\alpha^j}{j!} e^{-\alpha}
\]

\[
= \frac{\lambda}{\mu} \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} e^{-\alpha} - \frac{\lambda}{\mu} \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} e^{-\alpha}
\]

\[
= \frac{\lambda}{\mu} \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} e^{-\alpha} - \frac{\lambda}{\mu} \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} e^{-\alpha}
\]

That is,

\[
q = P(s, a) - \frac{\lambda}{\mu} P(s+1, a).
\]

Per unit time \(\lambda(1-q)\) \((\lambda-\delta)\) will enter service. The mean service time equals \(\mu^{-1}\). It follows that the carried load equals \(a^* = \lambda(1-q)\mu^{-1} = a(1-q)\). Thus, \(q = 1 - a^*/\alpha\).

Chapter 3, Exercise 13

'Suppose that a company with a private telephone network...'

Let \( s_1 \) number of flat rate trunks, \( s_2 \) number of measured rate trunks. Assume an ordered list such that a call will be carried by a flat rate trunk whenever possible. Evidently, this policy will minimize the relevant costs. The priority within the two classes of trunks is immaterial.

The associated hourly cost is

\[
H(s_1, s_2) = 14 s_1 + 30 \sum_{j=0}^{s_2} P_j
\]
(Chap. 3, Ex. 12)

where

$$\rho_j^* = a \left[ B(j-1, a) - B(j, a) \right],$$  \hspace{1cm} (3.18)

with \( a = 2 \) as long as

- If \( 14 > 30 \rho_j^* \), let \( j^* = 0 \).
- If \( 14 \leq 30 \rho_j^* \), let \( j^* \) be the maximal \( j \) such that \( 14 \leq 30 \rho_j^* \). By (3.18), \( \rho_j^* = 2.0 \times 10^{-4} \times 0.4400 = 0.5335 \) and \( \rho_2^* = 2.0 \times 10^{-4} \times 0.2186 = 0.3770 \). Hence, \( 14 < 30 \rho_j^* = 16.00 \), but \( 14 > 30 \rho_2^* = 11.37 \). As \( \rho_j^* > \rho_2^* \), obviously \( j^* = 2 \).

Considering the cost function \( H(s, s_2) \) and the relations \( \rho_j^* > \rho_2^* \),

$$e_j^* = \min (j^*, s) \hspace{1cm} (6)$$

is the optimal number of flat rate trunks out of a total of \( s (= s_1 + s_2) \) trunks.

It is a requirement that \( B(s_1 + s_2, 2) \leq 0.02 \). We have \( B(52) = 0.046 \) and \( B(6, 2) = 0.0121 \), and since the cost structure does not explicitly account for blocking costs, \( s_1 + s_2 = 6 \) is the optimal number of trunks. Hence, by (6),

$$e_1^* = \min (2, 6) = 2$$

is the optimal number of trunks, and the associated cost is

$$H(2, 4) = 14 + 2 \times 30 (\rho_2^* + \rho_2^* + \rho_2^* + \rho_2^*)$$
$$= 28 + 30a \left[ B(2, a) - B(4, a) \right] \hspace{1cm} \text{[by (3.18)]}$$
$$= 28 + 60 (0.4000 - 0.0121)$$
$$= 51.27.$$  

Given \( s_1 + s_2 = 6 \), the direct approach is to calculate \( H(s_2, s_2) \)

\( = 14s_2 + 60 (B(s_2, 2) - B(s_2, 2)) \) for \( s_1 = 0, 1, \ldots, 6 \). The result is:

<table>
<thead>
<tr>
<th>( s_1 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H(s_1, s_2) )</td>
<td>51.27</td>
<td>53.18</td>
<td>51.27</td>
<td>53.40</td>
<td>66.99</td>
<td>71.48</td>
<td>84.60</td>
</tr>
</tbody>
</table>

Again, \( e_1^* = 2 \).
Chapter 3, Exercise 14

Prove that in an Erlang loss system with ordered hunt...

Suppose there are s servers. Consider an arbitrary customer, denote by $A_j$ the event that on arrival he finds the first $j$ servers busy, and let $E_j$ denote the event that the customer will be served by server $j$, meaning that the first $j-1$ servers are busy, whereas server $j$ is free. Obviously, $A_j \subset A_{j-1}$ and $E_j = A_j - A_{j-1}$. Hence, $P(E_j) = P(A_j) - P(A_{j-1}) = P(A_j) (j=1, \ldots, s)$. With ordered hunt the first $j$ servers, $j \leq s$, function as an Erlang loss system, so $P(A_j) = B(j, a)$. Thus,

$$P(E_j) = B(j-1, a) - B(j, a) \quad (j=1, \ldots, s).$$

By (3.18),

$$\frac{P_j}{a} = B(j-1, a) - B(j, a) \quad (j=1, \ldots, s).$$

It follows that

$$P(E_j) = \frac{P_j}{a} \quad (j=1, \ldots, s).$$

Chapter 3, Exercise 15

Prove that the variance $\nu$ of the Erlang loss distribution...

We shall demonstrate that the variance of the state variable $J$ with distribution

$$P(J=j) = \frac{a^j}{\sum_{k=0}^{\infty} a^k k!} \quad (j=0,1,\ldots, s) \quad (33)$$

may be expressed as $\nu(J) = \alpha(1-\alpha)$.

First we prove the formula in the simple case $s=1$.

By (3.5),

$$E(J) = \sum_{j=0}^{\infty} j \frac{P_j}{a} = a^2 - a B(1, a).$$

$J$ is a zero-one variable, so that $E(J^2) = E(J)$. Hence,

$$E(J^2) = a - a B(1, a).$$
Hence, 

\[ V(J) = E^2(J) - E^2(J) = a \{ 1 - B(0, a) \} (1 - a [1 - B(0, a)]). \]

By (3.5), \( a = a \{ 1 - B(0, a) \}, \) and by (3.18), \( \rho_j = a \{ 1 - B(0, a) \}. \) Thus

\[ V(J) = v = a' [1 - \rho_j] \quad (s = 1). \]

Now consider the case \( s \geq 2. \) To begin, we express the variance \( V(J) \) in terms of \( s, a \) and \( \psi(\psi, a) \):

\[ E(J) = \sum_{j=1}^{s} \rho_j = a' = a - a \cdot B(s, a). \quad \text{(by (3.1))} \]

\[ E(J(J-1)) = \sum_{j=1}^{s} j \rho_j = \sum_{j=1}^{s} j \rho_j = \frac{a^2 \sum_{j=1}^{s} j \rho_j}{\sum_{j=1}^{s} \rho_j} = \frac{a^2 \sum_{j=1}^{s} j \rho_j}{\sum_{j=1}^{s} \rho_j} \]

\[ = a^2 \left[ 1 - \frac{1}{s} \sum_{j=1}^{s} \rho_j \right] = a^2 - a^2 B(s, a) - a B(s, a). \]

\[ E(J^2) = E(J(J-1)) + E(J) = a^2 - a^2 B(s, a) - a B(s, a) + a^2 B(s, a). \]

\[ E^2(J) = (a - a B(s, a))^2 = a^2 - 2a^2 B(s, a) + a^2 B(s, a) \]

Hence,

\[ V(J) = E(J^2) - E^2(J) = a - a B(s, a) - a B(s, a) + a^2 B(s, a) - a^2 B(s, a), \]

which can be rewritten as

\[ V(J) = a \{ 1 - B(s, a) \} (1 - a B(s, a)). \]

If the equation of Exercise 6 is solved with \( B(s-1, a) \) we find

\[ a B(s, a) = a B(s-1, a) \]

Thus,

\[ V(J) = a \{ 1 - B(s, a) \} (1 - a [B(s-1, a) - B(s, a)]). \]

Finally, using Equations (3.5) and (3.18) we obtain

\[ V(J) = v = a' [1 - \rho_j] \quad (s \geq 1). \]
Chapter 3, Exercise 16

Show that, for every integer \( s > a \), \( C(s, a) = \ldots \).

A rewriting of (4.8) gives

\[
C(s, a) = \frac{s^a \cdot \sum_{k=0}^{s-1} \frac{a^k}{k!}}{\sum_{k=0}^{s-1} \frac{a^k}{k!} + s - a(s-1)!} \quad (s > a).
\]

Dividing numerator and denominator by \( \sum_{k=0}^{s-1} a^k/k! \) and introducing \( B(s, a) = (a^s)/\sum_{k=0}^{s-1} a^k/k! \) we easily derive

\[
C(s, a) = \frac{sB(s, a)}{s - a(1 - B(s, a))} \quad (s > a). \tag{1}
\]

Another rewriting of (4.8) gives

\[
C(s, a) = \frac{s^{a-1} \cdot \sum_{k=0}^{s-1} \frac{a^k}{k!}}{\sum_{k=0}^{s-1} \frac{a^k}{k!} + s - a(s-1)!} \quad (s > a).
\]

Dividing numerator and denominator by \( \sum_{k=0}^{s-1} a^k/k! \) and introducing \( B(s-1, a) = (a^{s-1})/\sum_{k=0}^{s-1} a^k/k! \) we obtain

\[
C(s, a) = \frac{1}{1 + (s - a)[aB(s-1, a)]^s} \quad (s > a). \tag{2}
\]

By (1), for \( s-1 > a \), that is, for \( s > a+1 \),

\[
C(s-1, a) = \frac{(s-1)B(s-1, a)}{(s-1) - a(1 - B(s-1, a))} \quad (s > a+1).
\]

Solving for \( B(s-1, a) \) leads to

\[
B(s-1, a) = \frac{(s-1) - aC(s-1, a)}{s - aC(s-1, a)} \quad (s > a+1).
\]

Insertion of the above expression into (2) results in

\[
C(s, a) = \frac{1}{1 + (s - a)^s \sum_{k=0}^{s-1} \frac{a^k}{k!} (s - a)C(s-1, a)} \quad (s > a+1). \tag{3}
\]
Chapter 3, Exercise 17

'\textit{Review and reconsider Exercises 4 and 5 of Chapter 1.}'

The equilibrium state probabilities $[P_i]$ derived for the delay system in Exercise 4 and for the loss-delay system in Exercise 5 of Chapter 1, held for Poisson arrivals and exponential service times. This may be proved rigorously by modeling the systems as birth-and-death processes, and then applying Eq. (1).

For the loss-delay system of Exercise 5 of Chapter 1, let $n =$ number of servers, $b =$ waiting room size, $\lambda =$ arrival rate, $\mu =$ service rate, offered load $\bar{\lambda} = \lambda/\mu$. It was found that

$$\begin{align*}
P_i &= \begin{cases} 
\frac{a^i}{i!} P_0 & (i = 1, 2, \ldots) \\
\frac{a!}{i! (a+i)!} P_0 & (i = 0, 1, \ldots, n) 
\end{cases} \\
\text{where} \quad P_0 &= \left(\sum_{i=0}^{\infty} \frac{a^i}{i!} + \frac{(a\bar{\lambda})^n}{i=0} \right) \left(\frac{1 - (\frac{\mu}{\lambda})^{n+1}}{\lambda} \right), \quad P_0 ^{**} \\
\end{align*}$$

Denote by $P_L$, $P_W$, $P_S$, the equilibrium probabilities of being lost (denied service), having to wait in queue, and getting served immediately. Note, $\Pi_S = P_2$, since the arrival process is Poisson. Hence,

$$\begin{align*}
P_L &= \Pi_{n+1} = P_n, \\
P_W &= \sum_{j=1}^{n+1} \Pi_j = \sum_{j=0}^{\infty} \Pi_j \left(\frac{a^i}{i!} \right), \\
P_0 &= \sum_{i=0}^{\infty} \Pi_i = \sum_{j=0}^{n+1} P_j.
\end{align*}$$

By $(\ast)$,$$
\begin{align*}
P_L &= \frac{a^i}{i!} P_0, \\
P_W &= \frac{a^i}{i!} \left(1 - \frac{\lambda}{\mu} \right) P_0 \quad (n \geq 1), \\
P_S &= \sum_{j=0}^{n+1} \frac{a^i}{i!} P_j,
\end{align*}$$

with $P_0$ given by $(**)$.
Chapter 3, Exercise 18

"Is the analysis leading to (4.12) valid for..."

The answer is no. The reason is that the mean idle period, which is the mean residual interarrival time at the end of the busy period, is not, in general, equal to the mean interarrival time $\lambda^{-1}$ as in the case of Poisson arrivals, i.e. exponentially distributed interarrival times.

Chapter 3, Exercise 19

"Consider again the premise of Exercise 15. Now, however..."

The subject is an Erlang delay system with $s = 4$ trunks and $a = 2$. Let $s_1$ = number of flat-rate trunks, and let $s - s_1 = 4 - s_1$ = number of measured-rate trunks. $s_1$ must be set to minimize the offered unit hourly cost, assuming that flat-rate trunks have priority.

$$H(s_1) = 14s_1 + 30 \sum_{j = 1}^{s_0} \psi_j,$$

where

$$\psi_j = \hat{\psi}_j [1 - \rho C(s_0, a)] + \rho C(s_0, a), \quad (4.16)$$

and

$$\hat{\psi}_j = a [B(s_1, a) - B(s_0, a)]. \quad (3.18)$$

Here, $\rho = a/s = 0.5$, and $C(s_0, a) = C(4, 2) = 0.1739$ according to tables of the Erlang delay formula (4.8) see Fig. 4-3, Appendix A.

$\hat{\psi}_j$ is determined using tables of the Erlang loss formula (5.9).

We find

<table>
<thead>
<tr>
<th>$s_1$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_1$</td>
<td>0.6607</td>
<td>0.5398</td>
<td>0.5940</td>
<td>0.2536</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>0.6967</td>
<td>0.5790</td>
<td>0.5950</td>
<td>0.2976</td>
</tr>
</tbody>
</table>

Substitution of the $\psi_j$'s into (a) yields

<table>
<thead>
<tr>
<th>$s_1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(s_1)$</td>
<td>60.01</td>
<td>53.14</td>
<td>49.92</td>
<td>51.93</td>
<td>56.00</td>
</tr>
</tbody>
</table>

Best choice therefore is $s_1 = 2$, and $H(s_1) = 49.92$. \(\square\)
Chapter 3, Exercise 20

Prove that for an s-server Erlang delay system...

Special version of theorem

Let \( \{P^n_s\} \) be the equilibrium state probabilities of an Erlang loss system, and let \( \{P^n_s\} \) be the equilibrium state probabilities of an Erlang delay system. Suppose the systems have the same number of servers \( s \) and identical parameters \( \lambda \) and \( \mu \). In the delay system, let \( Q^n_j \) be defined as the conditional probability of being in state \( j \) given \( j \leq s \), that is, \( Q^n_j = P^n_j / \sum_{k=0}^{s} P^n_k \) \( (j=0, 1, ..., s) \). Then \( P^n_j = Q^n_j \) \( (j=0, 1, ..., s) \).

Proof. By (33),
\[
P^n_j = \frac{(\lambda/\mu)^j}{\sum_{k=0}^{s} (\lambda/\mu)^k} \quad (j = 0, 1, ..., s).
\]

By (34),
\[
P^n_j = \frac{(\lambda^* / \mu^*)^j}{P^n_0} \quad (j = 0, 1, ..., s).
\]

Assume that an equilibrium distribution exists, so that \( P^n_0 > 0 \). Then
\[
Q^n_j = \frac{P^n_j}{\sum_{k=0}^{s} P^n_k} = \frac{(\lambda/\mu)^j}{\sum_{k=0}^{s} (\lambda/\mu)^k} \quad (j = 0, 1, ..., s).
\]

Thus \( P^n_j = Q^n_j \) for \( j = 0, 1, ..., s \).

General version of theorem

Consider the birth-and-death processes with parameters \( \{\lambda^*_j, \mu^*_j\} \) and \( \{\lambda_j, \mu_j\} \). Assume equilibrium state distributions \( \{P^n_j\} \) and \( \{P^*_j\} \) exist and \( P^n_0 \geq 0, P^*_0 > 0 \). Assume \( \lambda^*_j = \lambda_j \) for \( j = 0, 1, ..., s-1 \) and \( \mu^*_j = \mu_j \) for \( j = 1, 2, ..., s \), for some \( s \geq 1 \). Let \( Q^n_j = P^n_j / \sum_{k=0}^{s} P^n_k \) and \( Q^*_j = P^*_j / \sum_{k=0}^{s} P^*_k \) \( (j = 0, 1, ..., s) \) be the conditional probability of being in state \( j \), given \( j \) \( s \). Then \( Q^n_j = Q^*_j \) for \( j = 0, 1, ..., s \).

Proof. The result follows easily from the fact that \( P^n_j / P^n_0 = P^*_j / P^*_0 = (\lambda^*_j / \mu^*_j) / (\lambda_j / \mu_j) \) \( (j = 1, 2, ..., s) \). Observe, in the special case above, \( \lambda^*_j = \mu^*_j \).

\[ \square \]
Chapter 3, Exercise 21

It will be shown that the statement made in Exercise 14 holds true also with "Erlang loss system replaced by "Erlang delay system."

We consider an Erlang delay system with $s$ servers and ordered input. Let $X$ be an arbitrary customer. Let $K$ be the state of the system when $X$ arrives. Let $D$ = \{ $X$ is delayed\}, $D$ = \{ $X$ is not delayed\}, $E_i$ = \{ $X$ is served by server $i$ \}. Then,

$$P(E_{i+1} | D) = P(E_i, D) + P(E_{i+1}, D).$$

First we calculate $P(E_{i+1}, D)$. Write $P(E_i, D) = P(E_i | D) P(D)$. Given Poisson arrivals, $P(D) = \frac{\sum_{n=0}^{s-1} P_n}{\sum_{n=0}^{s-1} P_n - s}$, and given exponential service times, $P(E_i | D) = 1/s$. Thus

$$P(E_{i+1}, D) = \frac{1}{s} C_s(a).$$

Next we calculate $P(E_i, D)$. Observe, \{ $E_i, D$ \} is equivalent to \{ $k \geq s$, $E_i$, $D$ \}. With Poisson traffic, $P(k \geq s) = \sum_{n=s}^{\infty} P_n$, and conditional on $k \geq s$, the probability of service by server $j$ without delay is $P_j / \alpha$, according to Exercise 14, since the system functions like an "Erlang loss system when $k \geq s$. Hence,

$$P(E_i, D) = P(k \geq s, E_i, D) = P(E_i | D) P(k \geq s) P(k \geq s),$$

$$= \frac{\alpha}{s} [1 - P(C_s(a))].$$

It follows that

$$P(E_i) = \frac{\alpha}{s} [1 - P(C_s(a))] + P(C_s(a))$$

where $P_j = \alpha [B(j-1, \alpha) - B(j, \alpha)]$. By (4.6) the numerator equals the load $P_j$ carried by the $j$th ordered server. Hence,

$$P(E_j) = \frac{\alpha}{s} \quad (j=1, 2, ..., s).$$

This result might have been easily derived by employing Little's theorem, $L = \lambda W$ (see Sec. 5.2), by which $P_j = \lambda P(E_j)$.
Chapter 3, Exercise 22

"Repeat Exercise 5 with "Erlang loss system" replaced by ..."

We consider an Erlang delay system with \( \lambda = 4 \) and \( \mu = 1 \). Then the offered load is \( a = \lambda \mu = 4 \). The objective function is \( H(a,c) = 4.25 - a P(W>0.5) 100 - sc \). Thus

\[
H(a,c) = 10 - 40 P(W>0.5) - sc.
\]

By (4.25), \( P(W > t) = C(a,y) e^{-a y - a t} \). Thus

\[
P(W>0.5) = C(a,4) e^{-a(4)-a(0.5)}
\]

It follows that

<table>
<thead>
<tr>
<th>( a )</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(a,4) )</td>
<td>0.044</td>
<td>0.040</td>
<td>0.038</td>
<td>0.036</td>
<td>0.034</td>
<td>0.032</td>
</tr>
<tr>
<td>( e^{-a(4)-a(0.5)} )</td>
<td>0.665</td>
<td>0.666</td>
<td>0.668</td>
<td>0.670</td>
<td>0.673</td>
<td>0.676</td>
</tr>
<tr>
<td>( P(W&gt;0.5) )</td>
<td>0.561</td>
<td>0.561</td>
<td>0.561</td>
<td>0.561</td>
<td>0.561</td>
<td>0.561</td>
</tr>
<tr>
<td>( H(a,c) )</td>
<td>-0.29</td>
<td>-0.19</td>
<td>-0.09</td>
<td>-0.01</td>
<td>0.09</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Thus, at \( a = 1.00 \), the optimal number of servers is 7, and the corresponding profit rate equals 0.80. The break-even point for \( a \) is \( c^* = 1.00 + 1.80/y = 1.26 \). Given the operating cost, the entrepreneur will break even for \( a = 7 \), but will have a negative profit rate for \( a > 7 \).

In case the entrepreneur may select any customer from the queue, the profit will be maximized for any \( a \), if the customer selected is the one who has waited the longest, but less than 1/4th.

Chapter 3, Exercise 23

"Consider a 10-server Erlang delay system that handles..."

| B C D | \( a \) | \( \lambda \) | \( \mu \) | \( a = \lambda \mu \) | \( C(a,\lambda) \) | \( E(W | W > a) \) |
|-------|-----|-----|-----|-----|-----|-----|
| case 0 | 10 | 10 | 10 | 100 | 0.105 | 8.4 \times 10^{-4} \times y/4 |
| case 1 | 10 | 11 | 1 | 11 | 0.109 | 2 \times Wd |
| case 2 | 10 | 11-\beta | 1 | 11-\beta | 0.109 | 2(1-\beta) \times Wd |
Note, by (9.12), \( E(W|W>0) = 1/(1-\rho) \phi \), where \( \phi = \alpha/\mu \). Thus, \( E(W|W>0) = \mu^{-1}/(\alpha - \mu) \).

The lesson is that \( C(\alpha, \mu) \) depends on \( s \) and \( a \), whereas the conditional mean wait \( E(W|W>0) \) depends on \( \alpha, \mu \) and \( \mu \). The response to a 1/2 increase in \( \lambda \) is a 100 \((0.0045 - 0.003)/0.003 = 30\% \) increase in \( C(\alpha, \mu) \) and a 100 \% increase in \( E(W|W>0) \). A 1/2 increase in \( \mu \), resulting in the same \( \alpha \), also leads to a 30\% increase in \( C(\alpha, \mu) \), but the increase in \( E(W|W>0) \) will be 100\%.

**Chapter 3, Exercise 24**

'In an Erlang delay system with service in order of arrival...'

By (9.24), \( P(W>0|W>0) = e^{-\lambda W} \), and by (9.26), \( E(W|W>0) = \frac{1}{\lambda} \).

Hence,

\[
P(W>E(W|W>0)|W>0) = e^{-\lambda E(W|W>0)} = e^{-\frac{1}{\lambda}} = 0.3679.
\]

**Chapter 3, Exercise 25**

'Consider a telephone system in which the central office...'

In an Erlang delay system with service in order of arrival the waiting-time distribution for blocked customers is the exponential distribution \( P(W>0|W>0) = e^{-\lambda W} \), see (9.24). Hence, if a customer has waited 30 sec, his remaining waiting time will still be exponentially distributed with mean \( C(\rho-\mu)^{-1} \).

One thing the customer should not do after waiting 30 sec is to put down the receiver and try again immediately. If he does that and waits until he gets through to the server, he will increase the waiting time by an expected 30 sec due to those customers who, thanks to his rash act, get ahead of him in the waiting line.

A better choice is to hang up and make another call T>0 sec later, waiting until served. The associated expected waiting time will converge to \( C(\rho-\mu)/\mu(\rho-\mu) \) as \( T \to \infty \). Since the limiting value is less than \( 1/\mu(\rho-\mu) \), the customer may be better off, everything considered, calling later.
'Show that in the Erlang delay system...

\[ E(W) = [1 - C(s, a)] E(W|W > 0) + C(s, a) E(W|W > 0) \]

\[ = C(s, a) E(W|W > 0). \]

By (4.24), with order-of-arrival service, the waiting time for blocked customers will be exponentially distributed with parameter \((1 - p)c\). Hence, \(E(W|W > 0) = 1/(c(1 - p))\), so that

\[ E(W) = \frac{C(s, a)}{(c(1 - p))}. \]

By (4.27),

\[ E^2(W) = \frac{C^2(s, a)}{(c(1 - p))^2}. \]

The variance is derived by substitution of these two expressions into \(V(W) = E(W^2) - E(W)^2\). The result is

\[ V(W) = \frac{1 - (1 - a)^2}{(c(1 - p))^2}. \]

\[ \text{Chapter 3, Exercise 27} \]

'Let \( W \) be the waiting time and \( T \) the sojourn time...'

\( s = 1 \). Hence, by (4.4) and (4.5), \( P_j = (1 - a)^j \). With Poisson arrivals \( \pi_j = P_j \), so the probability that an arbitrary customer finds \( j \) present in the system is

\[ \pi_j = (1 - a)^j \quad (j = 0, 1, \ldots). \]  

(1)

The probability that he will observe \( j \) in the queue, given that the server is occupied is \( P(q-j|W > 0) = \pi_j / \sum_{k=0}^{\infty} \pi_k = (1 - a)^j \). Hence, see also (4.23),

\[ P(q-j|W > 0) = (1 - a)^j \quad (j = 0, 1, \ldots). \]  

(2)

Now assume order-of-arrival service. The sojourn time will be the sum of \( j \) exponential service times, where the probability distribution of \( j \) (and therefore of \( j \) itself) is given by (1). The conditional waiting time is the sum of \( j \) exponential
service times where the probability distribution of $j$ (and therefore of $j^*$) is given by (2).

The two probability distributions (1) and (2), are identical. Consequently, the sojourn time and the conditional waiting time follow the same distribution in this case, namely

$$P(T > t) = P(W > t | W > 0) = e^{-\lambda t}$$

according to eq. (4.24).

Chapter 3, Exercise 28

Suppose $\alpha < \lambda$. For convenience, let $L_q$ and $W_q$ denote the mean queue length and mean waiting time, resp., and let $L_0$ and $W_0$ denote the mean number of customers in the system and mean sojourn time, resp.

By (4.4) and (47),

$$L_q = \sum_{k=0}^{\infty} (1 - p) p_k = \sum_{k=1}^{\infty} (1 - p) \frac{\alpha^k}{k!} \left( \frac{\alpha}{\lambda} \right)^{k-1}$$

$$= \frac{\alpha}{\lambda} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \left( \frac{\alpha}{\lambda} \right)^{k-1} = C(\alpha) \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \left( \frac{\alpha}{\lambda} \right)^{k-1}$$

The mean of a geometric distribution with parameter $p = 1 - \frac{\alpha}{\lambda}$ is $\frac{1-p}{p} = \frac{\alpha}{\lambda}$. Hence, $L_q = \frac{C(\alpha)}{\lambda - \alpha} \left( \frac{\alpha}{\lambda} \right)^{-1} = \frac{\lambda}{\lambda - \alpha}$. Substitution of this expression, and $\alpha = \beta$, yield

$$L_q = \lambda \left( \frac{\alpha}{\alpha - \beta} \right) \left( \frac{\alpha}{\lambda} \right)^{-1}$$

Finally, by (4.27),

$$L_q = \lambda W_q$$

Clearly, $L_q = L_0 + \frac{1}{\lambda}$ and $W_q = W_0 + \frac{1}{\lambda}$. Hence, using the relation $L_q = \lambda W_q$, it follows that

$$L_q = \lambda W_q$$
"Prove that in an Erlang delay system with order-of-arrival service."

Let \( P \) be the probability that a blocked customer will still be in the queue when next arrival takes place. Let \( \gamma_j \) be the probability that a blocked customer, who joins the queue when \( B-j \) customers are waiting, will still be in the queue at next arrival epoch. By the theorem of total probability,

\[
P = \sum_{j=0}^{\infty} \gamma_j P(G=j|W>0)
\]

The arrival rate is \( \lambda \) and, as long as all servers are busy, the service completion rate is \( \mu \). Therefore, by (9.23) of Chapter 4, \( \mu/\lambda \) is the probability, in all-busy states, that next event will be a service completion rather than an arrival. Since, with service in order of arrival, the blocked customer will get into service before next arrival if and only if at least \( j \) service completions occur before any arrival,

\[
\gamma_j = 1 - \left( \frac{\mu}{\mu + \lambda} \right)^j = 1 - \frac{1}{(1+\rho)^j},
\]

by (9.27), \( P(G=j|W>0) = (1-\rho)^j \rho^j \). Hence, \( \lambda/\mu \leq 1 \).

\[
P = \sum_{j=0}^{\infty} \left( 1 - \frac{1}{1+\rho} \right)^j \left( \frac{\mu}{\mu + \lambda} \right)^j = 1 - \frac{\mu}{\mu + \lambda} \frac{1}{1+\rho} = 1 - \frac{\mu}{\mu + \lambda} \frac{1}{1+\rho}.
\]

Hence, \( P = \rho \) as asserted.

"Let \( N \) be the number of customers found by an arrival."

Evidently, eq (9.14), \( P(W>x|W>0) = \sum_{j=0}^{\infty} P(W>x|N=j+1)P(G=j|W>0) \), holds for an Erlang delay system regardless of queue discipline. By definition, \( P(G=j|W>0) = P(N=j+1)/\sum_{k=0}^{\infty} P(N=k) \). For nonbiased queue disciplines \( \{N(t)\} \) is a birth-and-death process independent of the discipline. Hence \( P(N=k) \), and therefore \( P(G=j|W>0) \), are the same for all nonbiased q.d. By (9.23), for order-of-arrival service, \( P(G=j|W>0) = (1-\rho)^j \rho^j \). It follows that for all nonbiased q.d. we have \( P(W>x|N=k) = (1-\rho)^k \rho^j \). Substitution into (9.14) shows that \( P(W>x|W>0) = (1-\rho)\sum_{j=0}^{\infty} \rho^j P(W>x|N=j+1) \),

for all nonbiased queue disciplines.
Chapter 3, Exercise 31

Consider the differential-difference equations...

\[ \frac{d}{dt} F_j(t) = c F_{j+1}(t) - c F_j(t) \quad [t \geq 0; j = 0, 1, \ldots; F_0(t) = 0] \]  \hspace{1cm} (i)

where \( c \) is an arbitrary constant. Define

\[ F(x, t) = \sum_{j=0}^{\infty} F_j(t) x^j \]  \hspace{1cm} (ii)

\[ \sum_{j=0}^{\infty} \frac{d}{dt} F_j(t) x^j = c F_{j+1}(t) x^{j+1} - c F_j(t) x^j \quad (j = 0, 1, \ldots) \]

\[ \sum_{j=0}^{\infty} \frac{d}{dt} F_j(t) x^j = c x \sum_{j=0}^{\infty} F_{j+1}(t) x^j - c \sum_{j=0}^{\infty} F_j(t) x^j \]

\[ \frac{1}{dt} F(x, t) = \frac{1}{c} (x - 1) F(x, t) \]

Hence, \( F(x, t) = k(x) e^{-c(t-x)} \), by which

\[ F(x, t) = F(x, 0) e^{-c(t-x)} \]  \hspace{1cm} (iv)

a) In the case of a Poisson process Eq. (i) holds with \( c = \lambda \) and \( F_j(t) = P_j(t) = P(N(t) = j) \), according to Eq. (1.5) of Chapter 2. As \( F_0(0) = 1 \), clearly \( F(x, 0) = 1 \), so that in this case \( F(x, t) = e^{-\lambda(t-x)} e^{-\lambda t} \). By Eq. (iv) of Chapter 2, this is the generating function of a Poisson distribution with parameter \( \lambda \).

b) It follows that the probability of \( j \) arrivals in \( [0,t] \) equals

\[ F_j(t) = \frac{(\lambda t)^j}{j!} e^{-\lambda t} \quad (j = 0, 1, \ldots) \]

C) As long as there are at least \( k \) customers in the system, the departure process is Poisson with parameter \( \mu \). Let \( P_i(t) \) denote the probability of \( i \) departures within \( [0, t] \), assuming that all servers are busy. By the usual argument,

\[ P_i(t + h) = P_i(t) [1 - \mu h] + P_{i-1}(t) \mu h + o(h) \quad (i = 0, 1, \ldots) \]
(Chap. 3, Ex. 31c)

Hence, with \( P_{ij}(t+h) = 0 \),

\[
\sum_{i=1}^{k} P_{ij}(t+h) = [1 - \lambda x_i] \sum_{i=1}^{k} P_{ij}(t) + \mu x_i \sum_{i=1}^{k} P_{ji}(t) + o(h) \quad (j \neq 0, \ldots, k)
\]

Evidently, \( W_j(t) = \sum_{j=1}^{k} P_{ij}(t) \), so that

\[
W_j(t+h) = [1 - \lambda x_i] W_j(t) + \mu x_i W_j(t) + o(h) \quad (j \neq 0, \ldots, k),
\]

where \( W_j(0) = 0 \). Hence,

\[
\frac{d}{dt} W_j(t) = \mu x_i W_j(t) - \mu x_i W_j(t) \quad [t \geq 0, j \neq 0, \ldots, k, W_j(t+0)]
\]


Exercise (5) has the same form as Eq. (7). Consequently, if we define

\[
W(x,t) = \sum_{j=1}^{k} W_j(t)x_j,
\]

then, by (2) and (4),

\[
W(x,t) = W(x,0)e^{-\alpha x \mu t}.
\]

If \( W_j(0) = P(W > 0 | W = s) = 1 \) for all \( j \). Hence, for \( x < 1 \),

\[
W(x,0) = \sum_{j=1}^{k} x_j = \frac{1}{1 - x}.
\]

\[
P(W > t | W > 0) = (1 - \phi) \sum_{j=1}^{k} P(W > t | W = s_j) \phi^j \quad [\text{by Ex. 3a}]
\]

\[
= (1 - \phi) \sum_{j=1}^{k} W_j(t) \phi^j \quad [\text{by def. of } W_j(t)]
\]

\[
= (1 - \phi) W(t) \quad [\text{by def. of } W(x,t)]
\]

Equations (7), (8), (9) together yield

\[
P(W > t | W > 0) = (1 - \phi) \frac{1}{1 - \phi} e^{-\alpha x \mu t}
\]

\[
= e^{-\alpha x \mu t}.
\]
(Chap. 3, Ex. 31 b)

\[ h \] By Equations (6), (7) and (8),

\[
\sum_{j=0}^{\infty} W_j(t) x^j = \frac{1}{1 - x} e^{-\mu t} e^{\gamma t x}.
\]

Thus,

\[
\sum_{j=0}^{\infty} W_j(t) x^j = \left( \sum_{j=0}^{\infty} x^j \right) e^{-\mu t} \left( \sum_{k=0}^{\infty} \frac{\gamma k t^k x^k}{k!} \right),
\]

\[
\sum_{j=0}^{\infty} W_j(t) x^j = \frac{e^x}{1 - x} e^{-\mu t} \left( \sum_{k=0}^{\infty} \frac{(\gamma k t)^k x^k}{k!} \right).
\]

Equating coefficients of \( x^j \) on left- and right-hand sides yields

\[
W_j(t) = \frac{1}{\gamma t} \left( \frac{\mu}{\lambda} \right)^j \frac{\lambda^j}{j!} e^{-\mu t} \left( e^{-\gamma t} \right).
\]

(II)

---

*Chapter 3, Exercise 32*

'Service in random order.' - cf. Ex. 28 and 36 of Chap. 5

\[ W_i(t) = P(W \geq t | N = n + i) \]

(a) Let the test customer arrive at \( t = 0 \). During the time interval \([0, t]\) one of the following mutually exclusive events will occur:

1. The test customer departs from queue 1.
2. A customer arrives.
3. A customer other than the test customer departs from queue 1.
4. Neither arrival nor departure from queue (system) take place.
5. Two or more arrivals or departures occur.

Event 1 precludes the possibility that the test customer will be present in the queue at time \( t \) and, event 5 has probability \( a(t) \). Disregarding terms of order \( a(t) \), events 2, 3 and 4 have probability \( \lambda t (\gamma / \mu)^{\gamma} a(t) \) and \( 1 - (\lambda / \gamma) t a(t) \) respectively. Hence, by the theorem of total probability,

\[ W_i(t+\Delta t) = \lambda t W_i(\Delta t) + \int_0^{\Delta t} \lambda(t) a(t) W_i(\Delta t) + \int_0^{\Delta t} (\lambda + \mu) W_i(\Delta t) a(t) \mathrm{d}t \]

\[
\left[ t = 0, \ldots, W_i(t) = 0 \right].
\]

(1)

Hence,

\[ \frac{\mathrm{d}}{\mathrm{d}t} W_i(t) = \lambda W_{i+1}(t) + \int_0^t \lambda(t) a(t) W_i(t) \mathrm{d}t - (\lambda + \mu) W_i(t) \left[ \{ t = 0, \ldots, W_i(t) = 0 \} \right].
\]

(2)

where \( W_i(t) = 1 \) \( (t = 0, \ldots) \).
(Chap 3, Ex. 32 b)

Define

\[ W_j^{(0)} = \left. \frac{d}{\lambda x} W_j(t) \right|_{t=0} \quad (j = 0, 1, \ldots, v = 0, 1, \ldots). \]  

(3)

In particular, \( W_j^{(0)} = W_j(0) \neq 0 \).

Suppose that \( W_j(t) \) has the Maclaurin series representation

\[ W_j(t) = \sum_{v=0}^{\infty} \left( \frac{d^v}{\lambda x} \right)_v W_j^{(v)} \quad (j = 0, 1, \ldots). \]

(4)

According to Exercise 30,

\[ P(W > t | W > 0) = (1-\rho) \sum_{j=0}^{\infty} \rho^j W_j(t) \]

By (4),

\[ P(W > t | W > 0) = (1-\rho) \sum_{j=0}^{\infty} \rho^j \sum_{v=0}^{\infty} \left( \frac{d^v}{\lambda x} \right)_v W_j^{(v)} \]

\[ = (1-\rho) \sum_{v=0}^{\infty} \left( \frac{d^v}{\lambda x} \right)_v [1 + \sum_{j=0}^{\infty} \rho^j W_j^{(v)}] \]

Use of \( \sum_{j=0}^{\infty} \rho^j = (1-\rho)^{-1} \), and a change of the order of summation yield

\[ P(W > t | W > 0) = 1 + (1-\rho) \sum_{v=0}^{\infty} \frac{d^v}{\lambda x} \sum_{j=0}^{\infty} \rho^j W_j^{(v)}. \]

(5)

\[ \text{Repeated differentiation of Equation (2) gives} \]

\[ \frac{d^v}{\lambda x} W_j(t) = \lambda^{(v)} \frac{d^v}{\lambda x} W_j(t) + \frac{1}{v+1} \rho \frac{d^{v+1}}{\lambda x} W_j(t) \]

\[ - \left( \lambda + \nu \mu \right) \frac{d^v}{\lambda x} W_j(t) \quad [j = 0, 1, \ldots, v = 1, 2, \ldots] \]

Setting \( t = 0 \) we obtain

\[ W_0^{(v)} = \lambda W_0^{(v-1)} - (\lambda + \nu \mu) W_{v-1}^{(v-1)} \quad (v = 1, 2, \ldots), \]

\[ W_j^{(v)} = \lambda W_j^{(v-1)} + \frac{\nu}{v+1} \mu W_{j-1}^{(v-1)} - (\lambda + \nu \mu) W_j^{(v-1)} \quad (v = 0, 1, 2, \ldots, j = 1, 2, \ldots). \]

First we solve for \( v = 1 \). Recalling that \( W_j^{(0)} = 1 \) for all \( j \), we easily derive

\[ W_j^{(1)} = - \frac{\nu \mu}{v+1} \quad (j = 0, 1, \ldots). \]

(6)
(Chap. 3, Ex. 32 c)

Next we solve for \( n = 2 \), making use of (a). The result is

\[
W_i^{(2)} = \begin{cases} 
(\sigma \mu)^2 \left[ 1 + \frac{q}{2} \right] & (i = 0), \\
(\sigma \mu)^2 \frac{q}{(i+1)(i+2)} & (i \geq 1). 
\end{cases} \tag{\* \* \*}
\]

By (a),

\[
\sum_{i=0}^{\infty} q^i W_i^{(2)} = -\sigma \mu \sum_{i=0}^{\infty} \frac{q^i}{i+1} = -\sigma \mu \frac{q}{q-1}. \tag{\* \* \*}
\]

For \( 0 < q < 1 \), \( \sum_{i=1}^{\infty} i q^i = -\ln(1-q) = \ln \frac{1}{1-q} \). Hence,

\[
\sum_{i=1}^{\infty} i q^i W_i^{(2)} = -\sigma \mu \frac{q}{q-1} \ln \frac{1}{1-q}. \tag{\* \* \*}
\]

By (a **),

\[
\sum_{i=0}^{\infty} q^i W_i^{(2)} = (\sigma \mu)^2 \left[ 1 + \sum_{i=1}^{\infty} i q^i \frac{q^i}{i+1} \right] = (\sigma \mu)^2 \left[ 1 + \frac{q}{q-1} \ln \frac{1}{1-q} \right]. \tag{\* \* \*}
\]

Now let

\[
S(q) = \sum_{i=0}^{\infty} q^i W_i^{(2)}.
\]

Considering \( S(q) \) as a function of \( q \), differentiation results in \( dS(q)/dq = \sum_{i=0}^{\infty} i q^{i-1} = -\ln(1-q) \). Therefore, reversing the process, \( S(q) = -q \ln(1-q) + c \). From \( S(0) = 0 \) we derive \( c = 0 \). Thus

\[
\sum_{i=0}^{\infty} q^i W_i^{(2)} = q - (1-q) \ln \frac{1}{1-q}. \tag{\* \* \*}
\]

It follows that

\[
\sum_{i=0}^{\infty} q^i W_i^{(2)} = (\sigma \mu)^2 \left[ 2 - \frac{q}{q-1} \ln \frac{1}{1-q} \right]. \tag{\* \* \*}
\]

Finally, substitution of the found expressions for \( \sum_{i=0}^{\infty} q^i W_i^{(2)} \)

and \( \sum_{i=1}^{\infty} i q^i W_i^{(2)} \) into Equation (5) gives

\[
P(W > t | W > 0) = 1 - \sigma \mu \frac{q}{q-1} \ln \frac{1}{1-q} + \frac{(\sigma \mu)^2}{2} \left[ 2 - \frac{q}{q-1} \ln \frac{1}{1-q} \right] + \cdots \tag{\* \* \*}
\]

\( \square \)
Chapter 3, Exercise 33

Let $B(t)$ be the distribution function of the busy period.

It is clear that when service is in reverse order of arrival, then for all $N \geq 1$, the waiting time in a busy period initiated by the presently served customer (whose remaining time in service is exponentially distributed) and including all arrivals later than the first customer until he is permitted to enter service. (The same holds true for an $GI/M/\infty$ system for $N \geq 1$.) Hence, for an arbitrary customer, $P(W > t | W < 0) = B(t)$. It follows that the mean waiting time for waiting customers is the mean of the busy period, i.e., $E(W | W > 0) = b = \tau / (1 - p)$, by (4.18).

Chapter 3, Exercise 34

Show that $\lim \Pi^n[n] = \overline{P}_j$.

By (7.7) and $\hat{\delta} = \lambda / \mu$,

$$
\Pi^n[n] = \frac{\binom{n}{j} \hat{\delta}^j}{\sum_{k=0}^{\infty} \binom{k}{j} \hat{\delta}^j} \quad (j = 0, 1, \ldots, s).
$$

For $j = 0, 1, \ldots, s$,

$$
\lim_{n \to \infty} \frac{\binom{n}{j} \hat{\delta}^j}{\sum_{k=0}^{\infty} \binom{k}{j} \hat{\delta}^j} = \lim_{n \to \infty} \binom{n}{j} \hat{\delta}^j
$$

$$
= \left(\frac{\lambda}{\mu}\right)^j / j!
$$

Hence,

$$
\lim_{n \to \infty} \Pi[n] = \frac{\lim_{n \to \infty} \binom{n}{j} \hat{\delta}^j}{\sum_{k=0}^{\infty} \lim_{n \to \infty} \binom{k}{j} \hat{\delta}^j} = \frac{(\lambda / \mu)^j / j!}{\sum_{k=0}^{\infty} (\lambda / \mu)^k / k!} = \overline{P}_j,
$$

where, by Equation (3.3), $\overline{P}_j$ is the statistical equilibrium probability of $j$ busy servers in the Erlang loss system.
Chapter 3, Exercise 35

Four sources share access to two servers.

\( n \) = number of sources, \( n = 2 \); \( r^1 = 27 \) min.; \( r^2 = 3 \) min.; \( \lambda = 4 \) per hour.

The blocking probability is given by Engset's formula,

\[
\Pi_b[n] = \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha^k}{(1 + \frac{\lambda}{\mu})^{n+k}},
\]

where \( \alpha = \frac{\lambda}{\mu} \).

Since

\[
\Pi_b[4] = \frac{8(\frac{1}{2})^4}{1 + 3(\frac{1}{2}) + 3(\frac{1}{2})^2} = \frac{1}{32} = 0.03125,
\]

and

\[
\Pi_b[5] = \frac{6(\frac{1}{2})^5}{1 + 4(\frac{1}{2}) + 6(\frac{1}{2})^2} = \frac{2}{64} = 0.03125,
\]

the effect of going from four to five sources is a percent increase in the probability of blocking equal to

\[
P = 100 \left( \frac{\Pi_b[5]}{\Pi_b[4]} - 1 \right) = 80\%.
\]

To calculate the expected number of requests for some per hour, say \( r \), we go through the following steps:

(i) \( P_b[n] = \binom{n}{2} \frac{\alpha^2}{(1 - \frac{\lambda}{\mu})^{n+1}} \) (7.5),

(ii) \( a = a(1 - (1 - \frac{\lambda}{\mu}) P_b[n]) \), where \( \sigma = n\lambda(1 + \lambda) \) (7.8),

(iii) \( a = a(1 - P_b[n]) \) (7.9),

(iv) \( r = 60a/\mu^2 = 2a \).

For \( n = 4 \) and \( n = 5 \) we find

<table>
<thead>
<tr>
<th>( n )</th>
<th>( P_b[n] )</th>
<th>( a )</th>
<th>( a’ )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.03125</td>
<td>0.481</td>
<td>0.401</td>
<td>8.02</td>
</tr>
<tr>
<td>5</td>
<td>0.03125</td>
<td>0.4775</td>
<td>0.401</td>
<td>10.05</td>
</tr>
</tbody>
</table>

The lower bounds for \( r \) are 8 and 10, respectively.
Verify equation (8.10)

By Equations (6.8), (8.5), (8.6), (8.7) and (8.8),

\[
P(W > t) = \sum_{n=s+1}^{\infty} P(W > t | N = n) P(N = n)
\]

\[- \sum_{n=s+1}^{\infty} \left(e^{-\lambda t} \frac{\lambda^n}{n!} \right) \lambda^n \phi(n) \theta(t)
\]

\[- e^{-\lambda t} \sum_{n=s+1}^{\infty} \left(\frac{\lambda^n}{n!} \right) \frac{(n-1-s)!}{(n-s)!} \theta(t)
\]

\[- e^{-\lambda t} \sum_{n=s+1}^{\infty} \left(\frac{\lambda^n}{n!} \right) \frac{(n-1-s)!}{(n-s)!} \theta(t)
\]

where

\[
\phi(n) = \frac{s^n}{n!} + \alpha t
\]

and

\[
\theta(t) = \frac{t^n}{n!} + \beta t
\]

Thus

\[
P(W > t) = e^{-\alpha t} \sum_{n=s+1}^{\infty} \frac{\alpha^{n-1}}{n!} \frac{\beta^{n-s-1}}{(n-s)!} \theta(t)
\]

By the substitution \( k = n-1-s-j \),

\[
P(W > t) = e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \frac{\beta^{k}}{(k)!} \theta(t)
\]

Defining \( x = \alpha t \) and \( y = \beta t \) the double sum may be written

\[
\sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{y^k}{(k)!} = x y^n + \frac{x^{n+1}}{1!} \frac{y^n}{(n)!} + \frac{x^{n+2}}{2!} \frac{y^{n+1}}{(n+1)!}
\]

\[
\ldots + \frac{x^{n+k}}{k!} \frac{y^k}{(n+k)!}
\]

By the binomial formula,

\[
\sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{y^k}{(k)!} = \frac{(x+y)^n}{n!}.
\]

Hence

\[
\sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{y^k}{(k)!} = \frac{(x+y)^n}{n!}.
\]

We conclude that

\[
P(W > t) = e^{-\alpha t} \sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} e^{-\beta t}.
\]

(8.10)
Chapter 3, Exercise 3.7

"Verify equation (8.18) by direct calculation."

As \( u = a \) by (8.16), then
\[
\sum_{j=1}^{n} \lambda P[n] = \frac{\sum_{j=1}^{n} \lambda P[n]}{\lambda} + \sum_{j^{\alpha} = 1}^{n-1} (j-\lambda) P[n] = a + \frac{n}{\lambda} \sum_{j=0}^{n} (k+1) P[n].
\] (1)

By (8.3),
\[
P[n] = \frac{\lambda}{\mu} P[n-1].
\] (2)

By (1) and (2),
\[
\sum_{j=1}^{n} \lambda P[n] = a + \frac{n}{\lambda} \sum_{j=1}^{n} \frac{\lambda P[n]}{\mu} = \frac{\lambda}{\mu} \sum_{j=0}^{n} (k+1) P[n].
\] (3)

By (8.4),
\[
P[n] - \frac{\lambda}{\mu} P[n-1] = \frac{\lambda}{\mu} \sum_{j=1}^{n} (j-\lambda) P[n] = \frac{\lambda}{\mu} \sum_{j=1}^{n} \frac{(n-j)!}{j} \lambda (n-\lambda)! \frac{\lambda}{\mu} P[n].
\] (4)

By (8.3) this is seen to equal
\[
P[n] - \frac{\lambda}{\mu} P[n-1] = \frac{\lambda}{\mu} \sum_{j=1}^{n} (n-j) P[n] = \frac{\lambda}{\mu} \frac{(n-\lambda)!}{\lambda} \frac{\lambda}{\mu} P[n].
\]

Substituting \( n - \frac{\lambda}{\mu} \sum_{j=1}^{n} P[n] = \frac{\lambda}{\mu} \) from (8.17) we finally obtain
\[
P[n] = \frac{a}{\mu} P[n-1].
\] (5)

By (4) and (5),
\[
\sum_{j=1}^{n} \lambda P[n] = a (1 + \frac{E(W)}{\mu}).
\] (8.18) \( \square \)
Chapter 3, Exercise 38

'Reconsider Exercise 35, but instead of...

n = number of sources, a = 2, \( \gamma^{-1} = 27 \text{ min} \), \( \mu^{-1} = 3 \text{ min} \), \( a = \gamma / \mu = \frac{1}{4} \); blocked customers delayed.

To begin, we calculate the state distribution \( \{P_r[n]\}_{r=0}^{4} \), for \( n = 3, 4, 5 \), using Equations (8.3) and (8.4).

<table>
<thead>
<tr>
<th>( P_r[n] )</th>
<th>( r = 0 )</th>
<th>( r = 1 )</th>
<th>( r = 2 )</th>
<th>( r = 3 )</th>
<th>( r = 4 )</th>
<th>( r = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 3 )</td>
<td>0.7284</td>
<td>0.4778</td>
<td>0.3980</td>
<td>0.0015</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>0.4096</td>
<td>0.2910</td>
<td>0.1986</td>
<td>0.0089</td>
<td>0.0003</td>
<td>-</td>
</tr>
<tr>
<td>( n = 5 )</td>
<td>0.3875</td>
<td>0.3548</td>
<td>0.3135</td>
<td>0.0721</td>
<td>0.0013</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

(a) Blocking probability

\[ P_B = \sum_{r=0}^{\infty} P_r[n] = \sum_{r=0}^{\infty} P_r[n-1]. \]

\( n = 4 \):
\[ P_B = \sum_{r=2}^{\infty} P_r[4] = 0.0270 + 0.0015 = 0.0285 \]

\( n = 5 \):
\[ P_B = \sum_{r=3}^{\infty} P_r[5] = 0.0485 + 0.0054 + 0.0005 = 0.0544 \]

The percent increase in blocking probability is

\[ P = 100 \left( \frac{\sum_{r=2}^{\infty} P_r[5]/\sum_{r=2}^{\infty} P_r[4]}{\sum_{r=2}^{\infty} P_r[4]} - 1 \right) = 90\% \]

(b) Requests for service per hour

\[ \tau = 60 \frac{a}{\mu^{-1}} = 20a. \]

Since \( a = a' \) for a BCD system,

\[ a = a' = \sum_{r=0}^{\infty} r P_r[n] + \sum_{r=1}^{\infty} P_r[n] = P_r[n] + 2(1 - P_r[n] - P_r[n]) - 2 P_r[n] - P_r[n]. \]

\( n = 4 \):
\[ a = a' = 2 - 2 \cdot 0.06598 - 0.2910 - 0.3994 = 0.9988, \]
\[ \tau = 20a = 199.76. \]

\( n = 5 \):
\[ a = a' = 2 - 2 \cdot 0.3875 - 0.3548 - 0.3135 = 0.4986, \]
\[ \tau = 20a = 99.72. \]

Upper bounds for \( \tau \) are 8 and 10, respectively.
Server occupancy

\[ \phi = \alpha' \alpha \]

\[ \begin{align*}
\phi &= 0.3944/2 = 0.1972 \\
\phi &= 0.4986/2 = 0.2493
\end{align*} \]

Mean waiting time

By Eqs. (3.7), (8.9) and (8.12), the mean waiting time in seconds is

\[ E(W) = \frac{60}{\lambda} \sum_{i=0}^{\infty} (i+\phi) P_{i}[\lambda=1]. \]

\[ \begin{align*}
\lambda &= 4, \quad E(W) = 40(P[0]+2P[1]+4P[2]+6P[3]) = 0.00970 + 2.00018 = 2.010 sec \\
\end{align*} \]

By Eqs. (3.7), (8.16), (8.11) and (8.12),

\[ P(W > t) = c_0 e^{-\lambda t} \sum_{i=0}^{\infty} \left( \frac{\lambda t}{1+\phi} \right)^i \]

where

\[ \phi(t) = \frac{\lambda t}{1+\phi} + \phi \]

\[ \begin{align*}
c_0 &= P[0] \frac{(1+\phi)^2}{\lambda} \quad (8.11) \\
\phi(t) &= \frac{\lambda t}{1+\phi} + \phi \quad (8.12)
\end{align*} \]

with \( t \) measured in minutes.

For \( t = 3/11 \) min., \( e^{-\lambda t} = e^{-2} = 0.6065 \) and \( \phi(t) = 18.5 \). Also, \( \lambda = 1/14 \) and \( \phi = 1/18 \).

\[ \begin{align*}
\lambda &= 4, \quad c_0 = P[0] \frac{3!}{2!} \frac{1}{2} = 0.7286 \quad \text{and} \quad 0.00970 = 0.00970 \\
P(W > 3/11) &= c_0 e^{-2}(1+18.5) = 0.00970 \times 0.6065 \times 18.5 = 0.0177
\end{align*} \]

\[ \begin{align*}
\lambda &= 5, \quad c_0 = P[0] \frac{4!}{3!} \frac{1}{3} = 0.0078 \quad \text{and} \quad 0.00970 = 0.0078 \\
P(W > 3/11) &= c_0 e^{-2}(1+18.5+3^2) = 0.000999 \times 0.6065 \times 18.5 = 0.03496
\end{align*} \]
Proposition of time a source is idle.

Evidently,

\[ f = \frac{n - \sum_{k=1}^{n} k P_k}{n} \]

where

\[ n = 4: \quad f = \frac{27}{(27 + 2.7/60 + 3)} = 0.899 \]

\[ n = 5: \quad f = \frac{27}{(27 + 5.4/60 + 3)} = 0.897 \]

Compare with upper bound 0.9.

Chapter 3, Exercise 39

Using Equations (6.3) and (8.17), show that \( a \Pi_j[n] = (n-j) \delta_j[n] \).

Assume a BCD system with quasi-random input generated by \( n \) sources and with exponential service times. By Eq. (6.3),

\[ \Pi_j[n] = \frac{(n-j) P_j[n]}{\sum_{k=0}^{n-1} (n-k) P_k[n]} \quad (j = 0, 1, \ldots, n-1). \tag{6.3} \]

Clearly, \( \Pi_j[n] = 0 \). Thus (6.3) is valid also for \( j = n \). Furthermore, extending the summation to include \( k = n \) does not affect the value of the denominator. Hence, by (6.3),

\[ \Pi_j[n] = \frac{(n-j) P_j[n]}{n - \sum_{k=1}^{n} k P_k[n]} \quad (j = 0, 1, \ldots, n). \tag{*} \]

Now, by Eq. (8.17),

\[ n - \sum_{k=1}^{n} k P_k[n] = \frac{n}{a} \]

Inserting this expression into \( (*) \) we obtain

\[ a \Pi_j[n] = (n-j) \delta_j[n] \quad (j = 0, 1, \ldots, n). \]

\[ \square \]
Chapter 3, Exercise 40

'Consider a single-server queueing system with quasirandom.'

The queueing process under consideration is a birth-and-death process with $\lambda_1 = (n-1)\gamma$ and $\mu_j = n\mu$ for $j = 0, 1, \ldots, n$. Use of Eq. (1.1) results in the equilibrium state probabilities

$$P_i^n = \binom{n}{i} \delta^i \bar{P}_i^n \quad (i = 0, 1, \ldots, n),$$

with $\delta = \gamma/\mu$. Also,

$$A_1 = \sum_{i=0}^n P_i^n = (1+\delta)^n \bar{P}_1^n,$$

$$P_i^n = \frac{\binom{n}{i} \delta^i}{(1+\delta)^n} \quad (i = 0, 1, \ldots, n). \quad (\star)$$

Since we deal with a queue with quasirandom input and blocked customers delayed, it is true that $P_i^n = P_i^{n-1}$ for all $i = 0, 1, \ldots, n-1$. Hence, by (\star),

$$\bar{P}_i^n = \frac{\binom{n-1}{i} \delta^i}{(1+\delta)^{n-1}} \quad (i = 0, 1, \ldots, n-1).$$

Chapter 3, Exercise 41

'Queue with feedback'.

The arrival rate of new customers to the system is $\lambda$. The effective departure rate (from system) per customer in service is $(1-p)\mu$. Thus the queueing process is a birth-and-death process with the parameter $\lambda_\eta = \lambda$ for all $\eta$, and $\mu_\eta = n(1-p)\mu$ for $0 \leq n \leq \eta$, $\mu_\eta = 0$ for $n > \eta$. Offered load is $\alpha = \lambda/[1-(1-p)\mu]$.

The state of the system behaves precisely as in an ordinary $\text{BCD}$ queue with parameters $\lambda, \alpha$ and $(1-p)\mu$. Also, $\bar{P}_1 = \bar{P}_2$ due to Poisson arrivals, where $\bar{P}_1$ is the arrival distribution for new customers. The equilibrium probability that a new arrival finds all servers busy equals

$$C(s,\alpha) = \sum_{n=0}^{\infty} \bar{P}_n = \frac{\alpha^s}{\sum_{n=0}^{\infty} \alpha^n} \quad (\alpha < \delta),$$

with $C(s,\alpha)$ given by Erlang's delay formula, Eq. (48). □
Chapter 3, Exercise 42

'A single server serves customers of two priority classes.'

Poisson arrivals and exponential service times are assumed for both customer classes. The parameters are $\lambda_1$ and $\mu_1$ for the high priority class, $\lambda_2$ and $\mu_2$ for the low priority class. In parts (a)-(c) preemptive-repeat priority discipline will be assumed.

(a) By Eq. (5.13) of Chapter 2, the probability of preemption for a class 2 customer who has just entered or-reentered service equals $\lambda_1/(\lambda_1+\mu_2)$. Hence, the number $N$ of preemptions experienced by a class 2 customer has the geometric distribution

$$P(N = k) = \left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^k \frac{\mu_2}{\lambda_1+\mu_2} \quad (k = 0, 1, \ldots) \quad (1)$$

(b) The accumulated service time of a class 2 customer is not affected by preemptions (which interrupt the service). Given exponential service time and preemptive-repeat rule. Letting $S$ denote the total time an arbitrary class 2 customer occupies the server, we have

$$P(S \leq t) = 1 - e^{-\mu_2 t}, \quad (2)$$

just as if there were no preemptions allowed.

(c) Let $T$ denote the extended service time composed of the actual service time $S$ and the sum $\sum_{j=1}^N X_j$ of the $N$ time intervals during which the customer is preempted from service:

$$T = S + \sum_{j=1}^N X_j \quad (3)$$

Since $N$ and $\{X_j\}$ are independent, by part (b) of Exercise 4 of Chapter 2,

$$E(T) = E(S) + E(N)E(X) \quad (4)$$
(Chap. 3, Ex. 42.c)

where $E(X)$ denotes the common mean of $X_1, X_2, \ldots$. Now,

$$E(S) = \frac{1}{\lambda_0}, \quad (5)$$

$$E(N) = \frac{\lambda_0/\lambda_0}{\mu_0/(\lambda_0/\mu_0)} = \frac{\lambda_1}{\mu_2}, \quad (6)$$

$$E(X) = \frac{\mu_1}{1-(\lambda_1/\mu_1)} = \frac{1}{\mu_2-\lambda_1}. \quad (7)$$

Eq. (5) follows from Eq. (2). Eq. (6) follows from Eq. (1) since a variable with the geometric distribution $P(N=k) = q^{k-1}p$ has the mean $q/p$. See also Chapter 2, Exercise 21 a. (E(SN) - x(T).)

Eq. (7) follows from the observation that each $X_i$ is a busy period in a single-server queue with only class 1 customers. Thus Eq. (4/12) applies with $T = \mu_1$ and $a = \lambda_1/\mu_1$.

Substitution of (5), (6) and (7) into (4) yields

$$E(T) = \frac{\mu_1}{\mu_2(\mu_1-\lambda_1)} = \frac{\mu_1}{1-a_1}. \quad (8)$$

4. The service of high-priority customers is in no way affected by the presence of low-priority customers. Therefore, the waiting time $W_i$ of an arbitrary class 1 customer will have the distribution given by Eq. (4.25). Hence, by Exercise 4.b of Chapter 1,

$$P(W_i > t) = C(1, \lambda_1) e^{-\lambda_1 t} \frac{\lambda_1}{\lambda_0} e^{-(\mu_0-\lambda_1) t} \quad (9)$$

C. Conditions for bounded delays:

High-priority customers: $\frac{\lambda_1}{\mu_1} < 1$.

Low-priority customers: $\frac{\lambda_1}{\mu_1} + \frac{\lambda_0}{\mu_2} < 1$. \quad (10)

2. Under the exponential service-time assumption, the remaining service time at preemption will be exponentially distributed with mean $\mu_1$. Hence, an assumption of preemptive-resume priority discipline does not change the results in parts a - e.
Let $\lambda_E$ = eastbound traffic call rate, $\lambda_W$ = westbound traffic call rate, $\mu$ = service rate. Hence the offered loads are $a_1 = \lambda_E/\mu$ and $a_2 = \lambda_W/\mu$, respectively.

(a) Suppose $1 \le n \le s - 1$. A westbound call will be cleared in arrival state $j \ge s - n$, whereas an eastbound call will be cleared only if $j = s$.

The queueing process can be modeled as a birth-and-death process with $\lambda^+_j = \lambda_E + \lambda_W$ for $j = 0, 1, \ldots, s - n - 1$; $\lambda^+_j = \lambda_E$ for $j = s - n, \ldots, s - 1$; $\lambda^-_j = \mu$ for $j = 0, 1, \ldots, s$. By Eq. (3.15) of Chapter 2, then

\[
(\lambda^+_E + \lambda_W)P_0 = (s - n)\mu P_{n-1} \\
(\lambda^+_E + \lambda_W)P_{n-1} = (s - n)\mu P_n \\
\vdots \\
\lambda_E P_{s-n} = (s-n+1)\mu P_{s-n+1} \\
\vdots \\
\lambda_E P_{s-1} = s\mu P_s.
\]

(b) Recursive solution of the above state equations give

\[
P_j = \left\{
\begin{array}{ll}
\frac{(a_1 + a_2)^j P_0}{j!} & (j = 1, 2, \ldots, s-n), \\
\frac{(\frac{a_1 + a_2}{a_1})^{s-n} a_2^j P_0}{j!} & (j = s-n+1, \ldots, s).
\end{array}
\right.
\]

As usual, $P_s$ is found by use of the condition $\sum_{j=0}^{s} P_j = 1$.

(c) Loss on eastbound traffic = $P_0$.

Loss on westbound traffic = $\frac{\sum_{j=s-n+1}^{s} P_j}{s-n+1}$.
In order to minimize its telephone bill...

(a) Equilibrium state probabilities for flat-rate queue.

The equilibrium state probabilities \( \{P_\ell\} \) for the flat-rate queueing system can be found from the following equilibrium state equations:

\[
\begin{align*}
\lambda_1 P_0 + \lambda_2 P_1 &= 1, \\
\lambda_1 P_2 + \lambda_2 P_3 &= 0, \\
\lambda_2 P_4 &= 0, \\
\lambda_2 P_5 &= 0.
\end{align*}
\]

By recursive solution,

\[
P_j = \begin{cases} 
\frac{(a+q_j)^j}{\lambda_1} P_0 & (j = 1, 2, \ldots, s-1), \\
\frac{(a+q_j)^j}{a!} \frac{q_j}{\lambda_2} (1 - e^{-a/s}) P_0 & (j = s, s+1, \ldots),
\end{cases}
\]

and

\[
P_j = \left[ \sum_{k=0}^{s-1} \frac{(a+q_j)^k}{k!} \frac{q_j}{\lambda_2} (1 - e^{-a/s}) \right]^{-1}
\]

with \( a = \lambda_1 / \mu \) and \( q_j = \lambda_2 / \mu \), where \( q_j < s \) if \( q_j < s \), then \( P_j = 0 \) for all \( j \).

(b) The blocking probability \( B(s) = \sum_{j=s}^{\infty} P_j = \sum_{j=s}^{\infty} P_0 \sum_{i=0}^{\infty} \frac{(a+q_j)^i}{i!} \frac{q_j}{\lambda_2} (1 - e^{-a/s}) \)

\[
B(s) = \frac{a/(a+q_s)}{s} \frac{i}{e^{i/(a+q_s)} - 1/a} \quad (a < s).
\]

Observe that calculation of \( B(s) \) is facilitated by the formula:

\[
B(s) = s B(q_s) / (s - q_s U - B(q_s q_s)),
\]

as is easily verified, and the recurrence of Exercise 6 of Chapter 3.
(Chap. 3, Ex. 44 c.)

-69-

2 The overall cost per minute, \( c(s) \)

Cost parameters:
- \( c \) = cost per minute of a flat-rate trunk,
- \( \tau \) = cost of a toll call for the first minute of fraction thereof,
- \( \tau' \) = cost of a toll call for each additional minute of fraction thereof.

Letting \( M \) denote the random number of 1-minute intervals beyond the initial 1-minute interval, obviously

\[
c(s) = cs + \lambda B(s) [\tau + E(M)\tau'],
\]

since \( \lambda B(s) \) is the average overflow rate of high-priority customers requesting service from the flat-rate trunks, and \( \tau + E(M)\tau' \) is the mean cost of a toll call.

Now, given exponential service time with mean \( \mu \), the probability of holding the line for at least \( l \) more minutes equals \( e^{-\mu l} \) at the start of each 1-minute interval. Therefore, \( P(M=k) = (e^{-\mu})(1-e^{-\mu})^k \) for \( k = 0, 1, \ldots \). Hence \( E(M) = e^{-\mu}/(1-e^{-\mu}) \), and

\[
c(s) = cs + \lambda B(s) [\tau + e^{-\mu}/(1-e^{-\mu})] \cdot B(s).
\]

3 Mean waiting time for low-priority calls, \( E(W_L) \)

Let \( W_L \) = waiting time of an arbitrary low-priority customer. Observe \( P_{N=j} = B(s)(a_j/s_j)(1-a_j/s_j) \) for \( j = 0, 1, \ldots \), and \( \prod_{j=0}^\infty \) for \( N \) equal to the arrival state of the customer; \( E(W_L) = E(W_L|N=s_j) \cdot P_{N=s_j} \).

Hence, for \( a_j < s_j \), since \( P_{N=s_j} = P_{N=j} \),

\[
E(W_L) = \sum_{j=0}^\infty E(W_L|N=s_j) P_{N=s_j} = B(s)(s_j) \cdot \sum_{j=0}^\infty \left( \frac{a_j}{s_j} \right) \left( 1 - \frac{a_j}{s_j} \right) \cdot \frac{B(s)}{s_j - \lambda s} \quad \text{[analogous with Eq. 4.90]}
\]

4 Occupancy of flat-rate trunk, \( \rho \)

Clearly, \( \rho = 1 \) if \( a_j \geq s_j \). In case \( a_j < s_j \), the carried load on the flat-rate server group is \( \rho' = a_j [1-B(s)] + a_j \).

Hence,

\[
\rho = \frac{a_j}{s_j} = \frac{a_j [1-B(s)] + a_j}{s_j} \quad (a_j < s_j).
\]
Chapter 3, Exercise 45

'Time-varying Poisson input.'

(a) First assume that \( \lambda(t) \) is continuous and differentiable for all \( t \). By the reasoning used for derivation of Eq. (25) of Chapter 2 we find

\[
\frac{d}{dt} P_j(t) = \lambda(t) P_j(t) - \lambda(t) P_j(t) \quad (j = 0, 1, \ldots, P_j(t) - 0),
\]

the initial condition being \( P_j(0) = 1 \). Solution by recurrence starting with \( j = 0 \) yields the Poisson distribution

\[
P_j(t) = \frac{(\Lambda(t))^j}{j!} e^{-\Lambda(t)} \quad (j = 0, 1, \ldots)
\]

where

\[
\Lambda(t) = \int_0^t \lambda(x) dx
\]

The equations can be shown to hold also in the case of piecewise continuous and differentiable \( \lambda(t) \). This may be done by utilizing the additivity property of the Poisson distribution.

(b) In the infinite server queue a customer who arrives at time \( x < t \) will still be in service at time \( t \) with probability \( 1 - H(t-x) \). Hence, counting only arrivals that will be in the system at time \( t \), the effective arrival rate at \( x < t \) equals \( \lambda(x) = \lambda(1 - H(t-x)) \). The corresponding counting process is a Poisson process with time-varying rate. By (1) and (2) the number of customers in the system (in service) at \( t \) will have the Poisson distribution with mean

\[
\Lambda(t) = \int_0^t \lambda(1 - H(t-x)) dx = \int_0^t \lambda(x) [1 - H(t-x)] dx
\]

\[
= \Lambda(t) \left[ 1 - H(t) \right] + \int_0^t \lambda(x) dH(x)
\]

\[
= \lambda t \rho(t),
\]

where \( \rho(t) = 1 - H(t) + \int_0^t \lambda(x) dH(x) \). This proves Equation (3.11).

Finally, we observe that also Eq. (4.26) of Chapter 2 may be proved in a similar way by appeal to the notion of a time-varying Poisson process.
Chapter 3, Exercise 46

'Transient analysis of the single-server Erlang delay model.'

\[ a \] For the M/M/1 system, clearly

\[
P_i(t+h) = P_i(t)(1 - \lambda h) + P_i(t)\mu h + o(h),
\]

\[
P_i(t+h) = P_i(t)\mu h + P_i(t)(1 - (1+\mu)h) + P_{i+1}(t)\mu h + o(h) \quad (i = 1, 2, \ldots).
\]

Hence,

\[
\frac{d}{dt} P_i(t) = -\lambda P_i(t) + \mu P_i(t),
\]

\[
\frac{d}{dt} P_j(t) = \lambda P_{j-1}(t) - (\lambda + \mu) P_j(t) + \mu P_{j+1}(t) \quad (j = 1, 2, \ldots).
\]

Choosing \( \mu \) as the time unit, then \( \mu = 1 \) and \( \lambda = \lambda / \mu = \alpha \), so that the above equation system becomes

\[
\frac{d}{dt} P_0(t) = -\alpha P_0(t) + P_0(t), \quad (1)
\]

\[
\frac{d}{dt} P_j(t) = a P_{j-1}(t) - (1+a) P_j(t) + P_{j+1}(t) \quad (j = 1, 2, \ldots). \quad (2)
\]

\[ b \] Consider the auxiliary system of equations

\[
\frac{d}{dt} P_i(t) = a P_{i-1}(t) - (1+a) P_i(t) + P_{i+1}(t) \quad (j = 0, 1, 2, \ldots), \quad (3)
\]

\[
\frac{d}{dt} P_0(t) = a P_0(t) \quad (4)
\]

(3) and (4) together imply

\[
\frac{d}{dt} P_0(t) = -a P_0(t) + P_0(t), \quad (1a)
\]

and by (3),

\[
\frac{d}{dt} P_j(t) = a P_{j-1}(t) - (1+a) P_j(t) + P_{j+1}(t) \quad (j = 1, 2, \ldots) \quad (2a)
\]

Thus, if \( \hat{P}(t) \) \( (j = 0, 1, 2, \ldots) \) is a solution to Eqs. (3) and (4), then \( \hat{P}(t) \) \( (j = 0, 1, 2, \ldots) \) will be a solution to Equations (1a) and (2a).

As (1a) and (2a) are formally identical to (1) and (2), we conclude that if \( \hat{P}(t) \) for \( j = 0, 1, 2, \ldots \) solves (3) and (4), then \( \hat{P}(t) = \hat{P}(t) \) for \( j = 0, 1, 2, \ldots \) and \( t \), will also solve (1) and (2).
(Chap. 3, Ex. 46 e)

< Let \( \hat{P}_a(t) \) denote the generating function
\[
\hat{P}_a(t) = \sum_{n=0}^{\infty} \hat{P}_n(t) z^n.
\]
Multiplication of Eq. (5) by \( z^j \) and summation for all \( j \) result in
\[
\sum_{j=-\infty}^{\infty} d dz^n \hat{P}_n(t) z^j = a^n \sum_{j=-\infty}^{\infty} \hat{P}_n(t) z^j - (1+a) \sum_{j=-\infty}^{\infty} \hat{P}_n(t) z^j + a^n \sum_{j=-\infty}^{\infty} \hat{P}_n(t) z^j,
\]
\[
\frac{d}{dt} \sum_{j=-\infty}^{\infty} \hat{P}_n(t) z^j = [a^n - (1+a) + a^n] \sum_{j=-\infty}^{\infty} \hat{P}_n(t) z^j,
\]
which, by (6), is the same as
\[
\frac{d}{dt} \hat{P}_a(t) = [a^n - (1+a) + a^n] \hat{P}_a(t),
\]
whose general solution is
\[
\hat{P}_a(t) = G(a) e^{-[(1+a)t + (a^n + a^n)t]},
\]
where \( G(a) \) is any function of \( a \).

< Eq. (8) may be rewritten as
\[
\hat{P}(a, t) = G(a) e^{-[(1+a)t + (a^n + a^n)t]}.
\]
We shall use the fact that
\[
e^{\pm y(x+x^{-1})} = \sum_{n=-\infty}^{\infty} I_n(y) x^n,
\]
where \( I_n(y) \) are the modified Bessel functions. Now, setting \( y = 2a^n t \) and \( x = a^n t \), it is seen immediately that
\[
\hat{P}(a, t) = G(a) e^{-[(1+a)t + (a^n + a^n)t]} \sum_{n=-\infty}^{\infty} I_n(2a^n t) a^n x^{-n}. 
\]
Suppose \( G(a) \) has the expansion
\[
G(a) = \sum_{j=-\infty}^{\infty} c_j a^j.
\]

Insertion into (10) and collection of terms by powers of \( a \) lead to
\[
\hat{P}(a,t) = \sum_{j=-\infty}^{\infty} c_j a^j \sum_{k=-\infty}^{\infty} c_k a^{j+k} I_{j+k}(2a^m t).
\]

Comparison with Eq. (6) shows that
\[
\hat{P}_1(t) = e^{-i(\omega t)} \sum_{k=-\infty}^{\infty} c_k a^{j+k} I_{j+k}(2a^m t).
\]

For \( y = 0 \), Eq. (9) specializes to
\[
1 = \sum_{j=-\infty}^{\infty} I_j(0) \chi_j,
\]
whereby \( I_0(0) = 1 \) and \( I_k(0) = 0 \) for \( k \neq 0 \). By (12) then
\[
\hat{P}_1(0) = \sum_{j=-\infty}^{\infty} c_j a^j I_{j}(0) = c_{-i}.
\]

Let \( i \) be initial state, so that \( P_i(0) = 1 \). Then (provided \( \hat{P}_i(t) = \hat{P}_1(t) \) for \( j = 0, 1, \ldots \)) \( c_{-i} = 1 \), and \( c_k = 0 \) for \( k \neq 0 \) but \( k = -i \). Thus, Eq. (11) can be written
\[
\hat{P}_i(t) = e^{-i\omega t} \sum_{j=-\infty}^{\infty} c_j a^j I_{j}(2a^m t) + \sum_{k=-\infty}^{\infty} c_k a^{j+k} I_{j+k}(2a^m t),
\]

By Eq. (13),
\[
\hat{P}_g(0) = e^{-i(\omega t)} \left[ a^{-i} I_{-i}(2a^m t) + \sum_{k=-\infty}^{\infty} c_k a^{j+k} I_{j+k}(2a^m t) \right],
\]
\[
\hat{P}_e(0) = e^{-i(\omega t)} \left[ a^{-i} I_{-i}(2a^m t) + \sum_{k=-\infty}^{\infty} c_k a^{j+k} I_{j+k}(2a^m t) \right]
\]

From these expressions and Eq. (4), \( \hat{P}_g(0) = a \hat{P}_i(0) \), we obtain
\[
a^{-i} I_{-i} - \sum_{k=-\infty}^{\infty} c_k d_k I_{-i+k} = a^{-i} e^{-i(\omega t)} I_{-i(2a^m t)} + a^{-i} \sum_{k=-\infty}^{\infty} c_k d_k I_{-i+k},
\]

where \( d_k = c_k a^{j+k} \) and \( I_{-i} I_{-i}(2a^m t) \).
(Chap. 3, Ex. 46 h)

H Since $I_n(\text{q})=I_1(\text{q})$, Eq. (44) can be written

$$a^{-i\text{q}}I_1 + \sum_{k=1}^\infty d_k I_k = a^{-i\text{q}}I_1 + a^{2} \sum_{k=1}^\infty d_k I_k.$$  

This equation should hold for all $t$ and hence for all arguments $2\text{a}^2t$ of the $I_k$'s ($r=0,1,\ldots$). Consequently, the coefficient of each $I_r$ must equal zero. For example, for $r=2$ this requirement leads to the following set of equations,

$$
\begin{align*}
0 &= a^2 d_1 \\
0 &= a^2 d_2 \\
a^{-2i\text{q}}d_3 &= a^3 d_3 \\
a^{-2i\text{q}}d_4 &= a^3 d_4 \\
& \vdots \\
da^{-2i\text{q}}d_b &= a^b d_b
\end{align*}
$$

For arbitrary initial state $i$ the solution is

$$
\begin{align*}
d_{i,0} &= 0 \quad (k=1,2,\ldots, i) \quad \text{[hold if $i=0$]} \\
d_{i,1} &= a^{-i\text{q}} \\
d_{i,m} &= a^{-i\text{q}}-i[m-1] (1-a) \quad (m=1,2,\ldots)
\end{align*}
$$

H Substituting the found values of $d_i = a_i a_i^{2i}$ into (9), at the same time replacing $P_i(0)$ with $P_1(0)$ (the initialization permits this), we get

$$
\begin{align*}
P_i(t) &= e^{-i\text{q}t} \left[ a_{i} a_{i}^{-i\text{q}} I_{i,1}(2a^{2t}) + a_{i}^{-i\text{q}}-i I_{i,1+m}(2a^{2t}) + (1-a) \sum_{m=1}^{\infty} a_{i}^{-i\text{q}}-i[m-1] \frac{a_{i}}{1-a} I_{i,1+m}(2a^{2t}) \right].
\end{align*}
$$

Simplification results in

$$
\begin{align*}
P_i(t) &= a_{i}^{2i} e^{-i\text{q}t} \left[ I_{i,1}(2a^{2t}) + a_{i}^{-i\text{q}} I_{i,1+m}(2a^{2t}) + (1-a) \sum_{k=1}^{\infty} a_{i}^{-i\text{q}}-i[k-1] I_{i,1+k}(2a^{2t}) \right].
\end{align*}
$$

Equation (15) holds for all values of the carried load $a$. \(\blacksquare\)
Chapter 4, Exercise 1

Finite-source systems with nonidentical sources.

(a) Arguing as in Section 2 of Chapter 3 we find
\[ b_2 = \frac{\lambda_2 P(0)}{\lambda_2 P(0) + \lambda_1 P(0)} , \]
expressing that the probability that source 2 is blocked equals the number of blocked source 2-calls per unit time divided by the total number of source 2-calls per unit time. Hence
\[ b_2 = \frac{P(1,0)}{P(0,0) + P(1,0)} . \quad (1) \]

(b) With source 2 inactive, the system is in fact a one server, one source system, and the sole equilibrium state equation is \( \lambda_1 P(0) = \mu_1 P(1) \). Given \( P_0 + P_1 = 1 \), we find \( P = (\lambda_1/\mu_1)/(1 + \lambda_1/\mu_1) \). \( b_2 \) is defined as the probability that source 2 at a randomly chosen point in time finds the server occupied. Clearly,
\[ b_2 = P_0 \] That is, whether blocked customer cleared or delayed (i),
\[ b_2 = \frac{\lambda_1/\mu_1}{1 + (\lambda_1/\mu_1)} . \quad (2) \]

(c) Blocked customers cleared.
First we calculate source 2's blocking probability \( b_2 \). The conservation-of-flow equations when both sources are active are
\[
\begin{align*}
(\lambda_1 + \lambda_2) P(0,0) &= \mu_1 P(1,0) + \mu_2 P(0,1) \\
\mu_1 P(1,0) &= \lambda_1 P(0,0) \\
\mu_2 P(0,1) &= \lambda_2 P(0,0)
\end{align*}
\]
We need only \( P(1,0) \) in terms of \( P(0,0) \). The middle equation gives us \( P(1,0) = \lambda_1 P(0,0) \). By (1) then
\[ b_2 = \frac{(\lambda_1/\mu_1) P(0,0)}{P(0,0) + (\lambda_1/\mu_1) P(0,0)} = \frac{\lambda_1/\mu_1}{1 + (\lambda_1/\mu_1)} . \quad (3) \]
A comparison with Eq. (2) shows that in the BCC case \( b_2 = b'_2 \).
(Chap. 4, Ex. 1)

3. Blocked customers delayed

Again we calculate source 2’s blocking probability \( b_2 \). The conservation-of-flow equations are those found in Section 4.1 above. Omitting one equation we have

\[
\begin{align*}
(y_2 + m_1) P(0, 0) &= y_2 P(0, 0) + m_2 P(1, 0), \\
(y_1 + m_2) P(0, 1) &= y_1 P(0, 0) + m_1 P(1, 1), \\
m_1 P(1, 0) &= y_1 P(0, 1), \\
m_2 P(2, 1) &= y_2 P(0, 1).
\end{align*}
\]

Substituting the last two equations into the first two, and then eliminating \( P(0, 1) \) and solving for \( P(0, 0) \) we derive

\[
P(0, 0) = \frac{P(0, 0) k_1 (m_2 + y_2)}{m_1 m_2 + y_1 + y_2}.
\]

Substitution of this expression into Eq. (1) gives

\[
b_2 = \frac{m_1 m_2 + y_1 + y_2}{m_1 m_2 + m_1 y_1 + m_2 y_2 + y_1 + y_2}.
\]

or,

\[
b_2 = \frac{(y_1 m_2) y_2 + y_1}{(m_1 m_2) y_2 + (y_1 m_2) y_1 + y_1 + y_2}. \quad (4)
\]

We shall prove that \( b_2 = b'_2 \) if and only if \( m_1 = m_2 \). First assume \( m_1 = m_2 \). Then Eq. (4) reduces to \( b_2 = (y_1 m_2) y_1 / (m_1 m_2) y_2 \). Conversely, assume \( b_2 = b'_2 \). By Equations (2) and (4) this implies

\[
\frac{m_1 + y_2}{m_1 m_2 + m_1 y_1 + m_2 y_2 + y_1 + y_2} = \frac{m_1 + y_1}{m_1 + y_2} \quad [b_2 = b'_2].
\]

It follows easily that \( m_1 = m_2 \). We conclude that in this particular BCH model with nonidentical sources, the arriving customer’s 2-source distribution and his observer’s 1-source distribution are the same if and only if \( m_1 = m_2 \).
Chapter 4, Exercise 2

a. Three cities A, B, and C, are interconnected by two trunk groups.

In every case, let $\lambda_i, \mu_i, j_i$ denote arrival rate, completion rate, and number of calls in progress, respectively, for city connection no. $i$ ($i = 1, 2, 3$), where $i = 1$ refers to A-B, $i = 2$ refers to B-C, $i = 3$ refers to A-C. Let $P(j_1, j_2, j_3)$ be the equilibrium state probability of state $(j_1, j_2, j_3)$. Always, it is understood that $j_i, j_1, j_3 \geq 0$.

In case (a) the equilibrium state equations are:

\[
\begin{align*}
\lambda_1 \lambda_2 \lambda_3 \mu_1^* \mu_2^* \mu_3^* & P(j_1, j_2, j_3) = \\
\lambda_1 P(j_1-1, j_2, j_3) + \lambda_2 P(j_1, j_2-1, j_3) + \lambda_3 P(j_1, j_2, j_3-1) & \quad (j_2, j_3 \geq 0) \\
+ (j_1 \mu_1) P(j_1, j_2, j_3) + (j_2 \mu_2) P(j_1, j_2, j_3) + (j_3 \mu_3) P(j_1, j_2, j_3) & \\
\end{align*}
\]

\[
\begin{align*}
(\lambda_1 + j_1 \mu_1 + j_2 \mu_2 + j_3 \mu_3) P(j_1, j_2, j_3) = \\
\lambda_1 P(j_1-1, j_2, j_3) + \lambda_2 P(j_1, j_2-1, j_3) + \lambda_3 P(j_1, j_2, j_3-1) & \quad (j_1, j_2, j_3 \geq 0) \\
+ (j_1 \mu_1) P(j_1, j_2, j_3) + (j_2 \mu_2) P(j_1, j_2, j_3) + (j_3 \mu_3) P(j_1, j_2, j_3) & \\
\end{align*}
\]

\[
\begin{align*}
(\lambda_1 + j_1 \mu_1 + j_2 \mu_2 + j_3 \mu_3) P(j_1, j_2, j_3) = \\
\lambda_1 P(j_1-1, j_2, j_3) + \lambda_2 P(j_1, j_2-1, j_3) + \lambda_3 P(j_1, j_2, j_3-1) & \quad (j_1, j_2, j_3 \geq 0) \\
+ (j_1 \mu_1) P(j_1, j_2, j_3) + (j_2 \mu_2) P(j_1, j_2, j_3) + (j_3 \mu_3) P(j_1, j_2, j_3) & \\
\end{align*}
\]

If $s_i \to \infty$ and $s_3 \to \infty$, then $j_1, j_2$, and $j_3$ are independent Poisson variables, and

\[
P(j_1, j_2, j_3) = \frac{\lambda_1^{j_1}}{j_1!} \frac{\lambda_2^{j_2}}{j_2!} \frac{\lambda_3^{j_3}}{j_3!} e^{-\lambda} \quad [s_1 = \infty, s_2 = \infty].
\]

Because of a correspondence between terms on left and right-hand sides of all the equilibrium state equations in case (a), it is clear that also in the present case the solution has the form

\[
P(j_1, j_2, j_3) = \frac{\lambda_1^{j_1}}{j_1!} \frac{\lambda_2^{j_2}}{j_2!} \frac{\lambda_3^{j_3}}{j_3!} e^{-\lambda} \quad [s_1 = \infty, s_2 = \infty].
\]

Also, it is found from the normalization equation $\sum P(j_1, j_2, j_3) = 1$. 

In case (b) the equilibrium state equations are:

\[
\begin{align*}
\lambda_3 \lambda_4 + \lambda_5 \lambda_6 - \lambda_2 \lambda_7 + \lambda_3 \lambda_5 + \lambda_4 \lambda_6 - \lambda_1 \lambda_7 &= \left( \frac{j_1^*}{j_2^*} \max(0, j_2^* - s_1) \right) \\
\lambda_2 \lambda_3 + \lambda_4 \lambda_5 + \lambda_6 \lambda_7 - \lambda_1 \lambda_2 &= \left( \frac{j_1^*}{j_2^*} \max(0, j_2^* - s_2) \right)
\end{align*}
\]

\[
\begin{align*}
\lambda_7 \lambda_4 + \lambda_5 \lambda_6 - \lambda_2 \lambda_3 + \lambda_3 \lambda_6 + \lambda_4 \lambda_5 - \lambda_1 \lambda_4 &= \left( \frac{j_3^*}{j_4^*} \min(s_1, j_4^* - j_3^*) \right) \\
\lambda_4 \lambda_5 + \lambda_6 \lambda_7 - \lambda_2 \lambda_4 + \lambda_5 \lambda_7 + \lambda_6 \lambda_5 - \lambda_1 \lambda_6 &= \left( \frac{j_3^*}{j_4^*} \min(s_2, j_4^* - j_3^*) \right)
\end{align*}
\]

\[
\begin{align*}
\lambda_7 \lambda_5 + \lambda_6 \lambda_4 - \lambda_2 \lambda_7 + \lambda_3 \lambda_5 + \lambda_4 \lambda_6 - \lambda_1 \lambda_8 &= \left( \frac{j_3^*}{j_4^*} \min(s_1, j_4^* - j_3^*) \right) \\
\lambda_4 \lambda_7 + \lambda_5 \lambda_6 - \lambda_2 \lambda_4 + \lambda_6 \lambda_5 + \lambda_7 \lambda_4 - \lambda_1 \lambda_7 &= \left( \frac{j_3^*}{j_4^*} \min(s_2, j_4^* - j_3^*) \right)
\end{align*}
\]

\[
\begin{align*}
\lambda_2 \lambda_6 + \lambda_3 \lambda_5 - \lambda_1 \lambda_2 + \lambda_3 \lambda_6 + \lambda_4 \lambda_5 - \lambda_1 \lambda_3 &= \left( \frac{j_2^*}{j_3^*} \min(s_1, j_3^* - j_2^*) \right) \\
\lambda_1 \lambda_5 + \lambda_2 \lambda_6 - \lambda_1 \lambda_1 + \lambda_2 \lambda_5 + \lambda_3 \lambda_6 - \lambda_1 \lambda_2 &= \left( \frac{j_2^*}{j_3^*} \min(s_2, j_3^* - j_2^*) \right)
\end{align*}
\]

\[
\begin{align*}
\lambda_1 \lambda_7 + \lambda_2 \lambda_6 - \lambda_1 \lambda_1 + \lambda_2 \lambda_7 + \lambda_3 \lambda_6 - \lambda_1 \lambda_2 &= \left( \frac{j_1^*}{j_2^*} \min(s_1, j_2^* - j_1^*) \right) \\
\lambda_1 \lambda_5 + \lambda_2 \lambda_4 - \lambda_1 \lambda_1 + \lambda_2 \lambda_5 + \lambda_3 \lambda_4 - \lambda_1 \lambda_2 &= \left( \frac{j_1^*}{j_2^*} \min(s_2, j_2^* - j_1^*) \right)
\end{align*}
\]

The solution is of the same type as in case (a), namely

\[
P(j_1^*, j_2^*, s) = \frac{\lambda_1 \lambda_2}{j_1^*} \min(0, \frac{j_1}{j_1^*}) \times \frac{\lambda_3 \lambda_4}{j_2^*} \min(0, \frac{j_2}{j_2^*}) \times \frac{\lambda_5 \lambda_6}{j_3^*} \min(0, \frac{j_3}{j_3^*}) \times \frac{\lambda_7 \lambda_8}{j_4^*} \min(0, \frac{j_4}{j_4^*})
\]
(Chap 4, Ex 2 c)

Without a switching capability, the state description must be $(q_1, q_2, q_3, q_4)$ where $q_1$ and $q_2$ denote directly and indirectly connected calls between $A$ and $C$. This complicates the equilibrium state equations somewhat, but worse, the decomposition property is lost, so that the solution method above is not applicable.

In case (a), denote by $s_3$ the number of trunks directly connecting $A$ and $C$. Observe that $q_2 > s_3 \Rightarrow (q_2 < s_3, q_2 < s_3)$ implies similar implications of $q_2 > s_3$ and $q_4 > s_3$. The equilibrium state equations are:

\[
\begin{align*}
\lambda_1 & (\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) = (1 - \lambda_3)(\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) \\
\lambda_1 & (\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) + \lambda_3 P(q_1, q_2) (1 - s_3) + \lambda_4 P(q_1, q_2) (1 - s_3) \\
& + (\lambda_1)(\lambda_2 + \lambda_4) P(q_1, q_2) (1 - s_3) + (\lambda_1)(\lambda_2 + \lambda_4) P(q_1, q_2) (1 - s_3)
\end{align*}
\]

\[
\begin{align*}
\lambda_1 & (\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) = (1 - \lambda_3)(\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) \\
\lambda_1 & (\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) + \lambda_3 P(q_1, q_2) (1 - s_3) + \lambda_4 P(q_1, q_2) (1 - s_3) \\
& + (\lambda_1)(\lambda_2 + \lambda_4) P(q_1, q_2) (1 - s_3) + (\lambda_1)(\lambda_2 + \lambda_4) P(q_1, q_2) (1 - s_3)
\end{align*}
\]

\[
\begin{align*}
\lambda_1 & (\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) = (1 - \lambda_3)(\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) \\
\lambda_1 & (\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) + \lambda_3 P(q_1, q_2) (1 - s_3) + \lambda_4 P(q_1, q_2) (1 - s_3) \\
& + (\lambda_1)(\lambda_2 + \lambda_4) P(q_1, q_2) (1 - s_3) + (\lambda_1)(\lambda_2 + \lambda_4) P(q_1, q_2) (1 - s_3)
\end{align*}
\]

\[
\begin{align*}
\lambda_1 & (\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) = (1 - \lambda_3)(\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) \\
\lambda_1 & (\lambda_2 + \lambda_4) s_3 (1 - s_3) P(q_1, q_2) + \lambda_3 P(q_1, q_2) (1 - s_3) + \lambda_4 P(q_1, q_2) (1 - s_3) \\
& + (\lambda_1)(\lambda_2 + \lambda_4) P(q_1, q_2) (1 - s_3) + (\lambda_1)(\lambda_2 + \lambda_4) P(q_1, q_2) (1 - s_3)
\end{align*}
\]

Again, the solution is of the same type as in case (a), namely

\[P(q_1, q_2) = \frac{(\lambda_3)^{q_1}}{q_1!} \frac{(\lambda_4)^{q_2}}{q_2!} \frac{(\lambda_5)^{s_3}}{s_3!} c\]

for all feasible combinations $(q_1, q_2)$. As usual, $c$ is found from the condition $\sum P(q_1, q_2) = 1$. \(\square\)
A group of \( n \) trunks serves two types of Poisson traffic on a BCC.

Let \( j_1 \) be the number of ordinary calls and \( j_2 \) the number of wideband calls in progress. Then \( j_1 + k_2 \) is the total number of trunks that are held. The equilibrium state equations are as follows, assuming \( j_2 \geq 0 \) and \( k_2 \geq 0 \),

\[
\begin{align*}
(\lambda_1 + \lambda_2 + j_1 \mu_1 + j_2 \mu_2)P(j_1, k_2) &= (0 \leq j_1 + k_2 \leq n-
\lambda_1 P(j_1-1, k_2) + \lambda_2 P(j_1, k_2-1) \\
+j_1 \gamma_1 P(j_1-1, k_2) + j_2 \gamma_2 P(j_1, k_2+1) \\
(\lambda_1 + \lambda_2 + j_1 \mu_1 + j_2 \mu_2)P(j_1, k_2) &= (n-k_2 \leq j_1 + k_2 \leq n) \\
\lambda_1 P(j_1-1, k_2) + \lambda_2 P(j_1, k_2-1) \\
+j_1 \gamma_1 P(j_1-1, k_2) + j_2 \gamma_2 P(j_1, k_2+1) \\
(\lambda_1 + \lambda_2 + j_1 \mu_1 + j_2 \mu_2)P(j_1, k_2) &= (j_1 + k_2 = n) \\
\lambda_1 P(j_1-1, k_2) + \lambda_2 P(j_1, k_2-1).
\end{align*}
\]

By the same considerations as before, it is seen that the solution is given by

\[
P(j_1, k_2) = \frac{(\lambda_1 / \mu_1)^{j_1} (\lambda_2 / \mu_2)^{k_2}}{\lambda_1 \lambda_2 \gamma_1 \gamma_2} \left( \sum_{s=0}^{n-k_2} P(s) P(s) \right) ^{-1},
\]

where

\[
c = \left[ \sum_{s=0}^{n-k_2} \frac{(\lambda_1 / \mu_1)^{s} (\lambda_2 / \mu_2)^{n-s}}{\lambda_1 \lambda_2 \gamma_1 \gamma_2} \right]^{-1}.
\]

Denote by \( j_2^* = \left[ \frac{n-k_2}{\lambda_1 / \mu_1 + \lambda_2 / \mu_2} \right] \) the highest possible number of wideband calls in progress. Let \( P \) be the probability that an ordinary call is lost, and let \( P_w \) be the probability that a wideband call is lost. Then, clearly,

\[
P = P(j_1 + k_2 = 0) = \sum_{k_2=0}^{j_2^*} P(j_1, k_2),
\]

and

\[
P_w = P(s-k_2 \leq j_1 + k_2 \leq s) = \sum_{j_1=s-k_2}^{j_2^*} \sum_{k_2=s-j_1}^{j_2^*} P(j_1, k_2).
\]
Let $I(x) = 0$ if $x \leq 0$, $I(x) = 1$ if $x > 0$. With this notation, the equilibrium state equations can be written

\[
\sum_{i=1}^{n} \lambda_i I(k_i-j_i) + \sum_{i=1}^{n} \mu_i \Pi (j_i, k_i) = \left( \begin{array}{c} 0 \delta j_i k_i, i=1, \ldots, \mu_i \\ 0 \delta \sum_{i=1}^{n} j_i k_i, \sum_{i=1}^{n} j_i \end{array} \right)
\]

\[
+ \sum_{i=1}^{n} \mu_i I(k_i-j_i) \Pi (j_i, k_i) = \left( \begin{array}{c} 0 \delta j_i k_i, i=1, \ldots, \mu_i \\ 0 \delta \sum_{i=1}^{n} j_i k_i, \sum_{i=1}^{n} j_i \end{array} \right)
\]

where $j_i$ denotes the number of customers of type $i$, and $j_i \geq 0$.

The correspondence between LHS and RHS terms such as $\lambda_i I(k_i-j_i) \Pi (j_i, k_i)$ and $\mu_i I(k_i-j_i) \Pi (j_i, k_i)$ once more indicates a solution of the form

\[
\Pi (j_1, j_2, \ldots, j_n) = \left( \begin{array}{c} \lambda_1 j_1 \lambda_2 j_2 \cdots \lambda_n j_n \end{array} \right) \left( \begin{array}{c} \delta \sum_{i=1}^{n} j_i k_i, i=1, \ldots, \mu_i \\ \delta \sum_{i=1}^{n} j_i \end{array} \right)
\]

where, as usual, $\delta$ is determined from $\sum \Pi (j_1, j_2, \ldots, j_n)$.

Let $P_i$ be the equilibrium probability that all servers are busy, and let $P_i$ be the probability that $j_i = k_i$ while not all the servers are busy. Obviously,

\[P_i = \sum_{(j_1, j_2, \ldots, j_n)} \Pi (j_1, j_2, \ldots, j_n),
\]

where

\[S_0 = \{ (j_1, j_2, \ldots, j_n) : 0 \delta j_i k_i, i=1, \ldots, \mu_i \}, \]

and

\[P_i = \sum_{(j_1, j_2, \ldots, j_n)} \Pi (j_1, j_2, \ldots, j_n),
\]

where

\[S_i = \{ (j_1, j_2, \ldots, j_n) : j_i = k_i, 0 \delta j_i k_i, i=1, \ldots, \mu_i \}.
\]

By the assumption of Poisson arrival streams, the probability

\[P_i \text{ that a customer of type } i \text{ will be blocked equals}
\]

\[P_i = P_0 + P_i.
\]
Chapter 4, Exercise 5

'A group of 2 serves two types of customers on a 2CC basis.'

(a) The equilibrium state equations are, for \( j_1, j_2 \geq 0, \)

\[
\begin{align*}
(\lambda + (n-1)\mu + j_1\mu + j_2\mu) P_n(j_1, j_2) &= \lambda P_{n-1}(j_1-1, j_2) + (n-1)\mu P_{n-1}(j_1, j_2) + j_1\mu P_{n+1}(j_1-1, j_2) + j_2\mu P_{n+1}(j_1, j_2) \\
(\lambda + j_1\mu + j_2\mu) P_n(j_1-1, j_2) &= \lambda P_{n+1}(j_1-1, j_2) + (j_1\mu + j_2\mu) P_n(j_1, j_2-1) \\
(\lambda + j_1\mu + j_2\mu) P_n(j_1, j_2-1) &= \lambda P_{n+1}(j_1, j_2-1) + (j_1\mu + j_2\mu) P_n(j_1, j_2)
\end{align*}
\]

In case \( a = \infty, \) the equilibrium states \( j_1 \) and \( j_2 \) are independent, \( \lambda P_n(j_1) = \mu P_n(j_1+1) + (n-1)\mu P_n(j_1) + j_1\mu P_{n+1}(j_1) + j_2\mu P_{n+1}(j_1), \) which can be decomposed into equations of the two types \( \lambda P_n(j_1) = (\mu + (n-1)\mu + j_1\mu) P_{n-1}(j_1-1) \) and \( (\mu + j_1\mu + j_2\mu) P_n(j_1, j_2-1) = (\lambda + j_1\mu + j_2\mu) P_n(j_1, j_2) \) which both hold equally by \((b). \)

(b) Customers of type 1 arrive in a Poisson stream. Therefore, \( \Pi^1_{n}(j_1, j_2) = P_n(j_1, j_2) \) \( (0 \leq j_1, j_2 \leq \infty) \) and

\[
\Pi^2_{n}(j_1, j_2) = P_n(j_1, j_2) \quad (0 \leq j_1, j_2 \leq \infty)
\]
\( (\text{Chap. 24, Ex. 5}) \)

\[
\Pi_n^R(y_{1/2}) = \frac{(n-1)!}{\sum_{k=0}^{n-1} x_k} \left( \frac{(n-2)!}{\sum_{k=0}^{n-2} x_k} \right) P_n(y_{1/2}) \quad (0 \leq y_{1/2} \leq s_0, s_1, s_2)
\]

By (i), then,

\[
\Pi_n^L(y_{1/2}) = \frac{(n-1)!}{\sum_{k=0}^{n-1} x_k} \left( \frac{(n-2)!}{\sum_{k=0}^{n-2} x_k} \right) P_n(y_{1/2}) \quad (0 \leq y_{1/2} \leq s_0, s_1, s_2)
\]

We shall deal with the two cases \( n \leq s \) and \( n > s \) in turn.

\( n \leq s \). Here \( \min(s, n) = n \). Obviously, \( \Pi_n^R(y_{1/2}) = P_n(y_{1/2}) = 0 \).

For \( \frac{s}{2} \leq n - i = \min(s, n - i) \), Eq. (a) gives, after rewriting,

\[
\Pi_n^L(y_{1/2}) = \frac{(n-1)!}{\sum_{k=0}^{n-1} x_k} \left( \frac{(n-2)!}{\sum_{k=0}^{n-2} x_k} \right) P_n(y_{1/2}) \quad (0 \leq y_{1/2} \leq s_0, s_1, s_2)
\]

Comparison with Eq. (i) shows that \( \Pi_n^L(y_{1/2}) = P_n(y_{1/2}) \) for \( 0 \leq y_{1/2} \leq s_0 \) and \( \frac{s}{2} \leq \min(s, n) \). Thus we have shown that if \( n \leq s \), then \( \Pi_n^L(y_{1/2}) = P_n(y_{1/2}) \) for \( 0 \leq y_{1/2} \leq s_0 \) and \( \frac{s}{2} \leq \min(s, n) \).

\( n > s \). Here \( \min(s, n) = \min(s, n - 1) \) \((\ast)\) and \( n - \frac{s}{2} > 0 \) for all feasible \( y_{1/2} \). By (a) it follows that, again,

\[
\Pi_n^L(y_{1/2}) = \frac{(n-1)!}{\sum_{k=0}^{n-1} x_k} \left( \frac{(n-2)!}{\sum_{k=0}^{n-2} x_k} \right) P_n(y_{1/2}) \quad (0 \leq y_{1/2} \leq s_0, s_1, s_2)
\]

Comparison with Eq. (i) shows that also if \( n > s \), then \( \Pi_n^L(y_{1/2}) = P_n(y_{1/2}) \) for \( 0 \leq y_{1/2} \leq s_0 \) and \( \frac{s}{2} \leq \min(s, n) \). Note that despite appearances, the range of \( y_{1/2} \) is not the same for \( \Pi_n^L(y_{1/2}) \) when \( n \leq s \) and when \( n > s \). Since, in the latter case, we have just made the substitution \( \min(s, n) = \min(s, n - 1) \), we conclude that, for any \( n \geq 1 \),

\[
\Pi_n^L(y_{1/2}) = P_n(y_{1/2}) \quad (0 \leq y_{1/2} \leq s_0, s_1, s_2)
\]
For \( j_1, j_2 \geq 0 \), the equilibrium state equations are:

\[
\begin{align*}
\left( \frac{n_{j_1} - j_1}{n_1 + n_{j_1 - j_1}} \lambda + \frac{n_{j_2} - j_2}{n_2 + n_{j_2 - j_2}} \lambda + j_1 \mu + j_2 \mu \right) P(j_1, j_2) &= \quad (j_1 \leq n_1, j_2 \leq n_2) \\
\frac{n_{j_1} - j_1}{n_1 + n_{j_1 - j_1}} \lambda P(j_1 + 1, j_2) + \frac{n_{j_2} - j_2}{n_2 + n_{j_2 - j_2}} \lambda P(j_1, j_2 - 1) + (j_1 + 1) \mu P(j_1 + 1, j_2) + (j_2 - 1) \mu P(j_1, j_2 - 1) &\quad (j_1 = n_1, j_2 < n_2) \\
\frac{n_{j_1} - j_1}{n_1 + n_{j_1 - j_1}} \lambda P(j_1 + 1, j_2) + \frac{n_{j_2} - j_2}{n_2 + n_{j_2 - j_2}} \lambda P(j_1, j_2 - 1) + (j_1 + 1) \mu P(j_1 + 1, j_2) &\quad (j_2 = n_2, j_1 < n_1) \\
\frac{n_{j_1} - j_1}{n_1 + n_{j_1 - j_1}} \lambda P(j_1 - 1, j_2) + \frac{n_{j_2} - j_2}{n_2 + n_{j_2 - j_2}} \lambda P(j_1, j_2 + 1) &\quad (j_1 = n_1, j_2 = n_2) \\
\frac{n_{j_1} - j_1}{n_1 + n_{j_1 - j_1}} \lambda P(j_1 - 1, j_2) + \frac{n_{j_2} - j_2}{n_2 + n_{j_2 - j_2}} \lambda P(j_1, j_2 + 1) + (j_1 - 1) \mu P(j_1 - 1, j_2) + (j_2 + 1) \mu P(j_1, j_2 + 1) &\quad (j_1 < n_1, j_2 = n_2)
\end{align*}
\]

Clearly, a solution to the equations

\[
\begin{align*}
\frac{n_{j_1} - j_1}{n_1 + n_{j_1 - j_1}} \lambda P(j_1, j_2) &= \quad (j_1 \leq n_1, j_2 \leq n_2) \\
\frac{n_{j_1} - j_1}{n_1 + n_{j_1 - j_1}} \lambda P(j_1 + 1, j_2) + (j_1 + 1) \mu P(j_1 + 1, j_2) &\quad (j_1 = n_1, j_2 < n_2) \\
\frac{n_{j_1} - j_1}{n_1 + n_{j_1 - j_1}} \lambda P(j_1 - 1, j_2) + (j_1 - 1) \mu P(j_1 - 1, j_2) &\quad (j_1 = n_1, j_2 > n_2)
\end{align*}
\]

will also be a solution to the equilibrium state equations above.
By recursive solution, (4) yields
\[ P(j,1) = \frac{n_j(n_j + n_{j-1} - 1)!}{n_j!(n_j + n_{j-1})(n_j + n_{j-2})} P(0,j) \quad (j = 0, \ldots, n, s) \]
and (5) yields
\[ P(j,2) = \frac{n_j(n_j + n_{j-1} - 1)!}{n_j!(n_j + n_{j-1})(n_j + n_{j-2})} P(0,j) \quad (j = 0, \ldots, n) \]
Combination of the two equations and the substitution \( \frac{n_j}{n} = a \) results in
\[ P(j,1) = \frac{\binom{n}{j}}{\binom{n+j}{j}} \cdot \frac{n_{j-1}}{(j + 1)!} P(0,j) \quad (0 \leq j \leq n) \]

When \( (n, m) = (s, s) \) no call is lost if any trunk is idle, and every call is lost if all trunks are occupied. By (5) of Chapter 3 then \( P(0,1) = \left[ \frac{n_{j-1}}{(j + 1)!} P(0,j) \right] \).
Furthermore, in this case, an arriving customer will be directed to every idle trunk. Consequently, the \( j + 1 \) occupied trunks are actually drawn at random from the \( n_{j-1} \) trunks. By the hypergeometric distribution, \( P(j,1) = \binom{n_j}{j}/\binom{n+j}{j} \) follows. Thus, for \( (n, m) = (s, s) \), \( P(j,2) = P(j,1)^s \), in agreement with the derived formula for \( P(n, s) \). This may suggest that the formula holds also for \( n \leq s \), and \( n \geq \frac{s}{s} \), respectively.

Chapter 4, Exercise 7

"Customers arrive according to a Poisson process..."

4. By the theorem of total probability,
\[ \sum_{x_1 + \cdots + x_s = n} P(x_1, x_2, \ldots, x_s) = P(x) \quad (j = 0, \ldots, s) \]
for any \( \{x_i\} \). If \( M_1 = \ldots = M_s = \mu \), then the model specializes to the Erlang loss model, with \( P_j \) given by Eq. (5) of Chapter 3.
(Chap. 4, Ex. 7a.)

With random server selection and \( \mu_1 = \ldots = \mu_K \), clearly all combinations of \( j \) busy servers have equal probability. Since the number of combinations is \( \binom{N}{j} \),

\[
\mathbb{P}(x_{i-1}, x_j) = \binom{N}{j}^{-1} \mathbb{P}_j \quad (x_i \ldots x_j; \mu_1 = \ldots = \mu_K)
\]

\( \blacklozenge \) For \( x_i = 0, i \), the equilibrium state equations may be written:

\[
\begin{aligned}
&\left( \lambda \sum_{j=1}^{K} \frac{\mathbb{P}_j}{x_j} (1-x_j) \right) \mathbb{P}(x_1, \ldots, x_n) = \left( 0 \leq x_i \leq 1 \right) \\
&\lambda \sum_{j=1}^{K} \left( \mathbb{P}(x_1, x_j, \ldots, x_n) + \mathbb{P}(x_1, x_j, \ldots, x_n) + \ldots + \mathbb{P}(x_1, x_j, \ldots, x_n) \right)
\end{aligned}
\]

\[
\begin{aligned}
+ (x_i+x_j) \mathbb{P}(x_i, x_j, \ldots, x_n) + \ldots + (x_i+x_n) \mathbb{P}(x_i, x_j, \ldots, x_n) + (x_n+x_{n-1}) \mathbb{P}(x_1, x_2, \ldots, x_n)
\end{aligned}
\]

\[
\mathbb{P}(x_{i-1}, x_i) = \binom{N}{j}^{-1} \mathbb{P}_j (x_i \ldots x_j)
\]

Once more, there is a pairwise correspondence between LHS and RHS terms. It is seen that the equilibrium state equations will be satisfied by \( \mathbb{P}(x_1, \ldots, x_n) \) satisfying

\[
\mathbb{P}(x_1, \ldots, x_n) = \mathbb{P}(0,0,\ldots,0)
\]

Let \( \mathbb{P}_0 = \mathbb{P}(0,0,\ldots,0) \). By (\( \ast \)), \( \mathbb{P}(x_i, x_j, \ldots, x_n) = \frac{\lambda}{\mathbb{P}_0} \mathbb{P}_0 \) for \( x_i = 0 \) and \( x_j = j^{-1} \) (i.e. \( x_i = 0 \) for \( x_i \)). This finding can be expressed:

\[
\begin{aligned}
\mathbb{P}(x_i, x_1) = \frac{\mathbb{P}(x_i, x_1)}{x_i} \mathbb{P}_0
\end{aligned}
\]

By recursion, (\( \ast \)) yields

\[
\begin{aligned}
\mathbb{P}(x_i, x_1) = \frac{\mathbb{P}(x_i, x_1)}{x_i} \mathbb{P}_0
\end{aligned}
\]

\[
(x_j - j^{-1}; x_i = j)
\]

Notice, the formula also holds for \( x_i = 0, j \). Finally, rewriting this formula we obtain

\[
\begin{aligned}
\mathbb{P}(x_1, \ldots, x_n) = (\frac{\lambda}{\mathbb{P}_0})^{-1} \mathbb{P}(x_1, \ldots, x_n)
\end{aligned}
\]

As usual, \( \mathbb{P}_0 \) is found by the normalization condition.
Chapter 4, Exercise 8

Network of queues.

a Let \( \lambda_i \) denote the mean arrival rate at \( Q_i \). Obviously,

\[
\lambda_1 = \lambda, \quad \lambda_2 = P_2 \lambda, \quad \lambda_3 = P_3 \lambda, \quad \lambda_4 = (1 - P_2) \lambda.
\]

In the following, it will be assumed that \( \lambda_i / \mu_i < \epsilon_i \) for all \( i \).

The arrival process at \( Q_i \) is Poisson. By Burke's theorem, then, the equilibrium output from \( Q_i \) is Poisson. The assignment of this output by lottery leads to a decomposition into independent Poisson streams, with rates \( \lambda_2 = P_2 \lambda \) and \( \lambda_3 = P_3 \lambda \), respectively. We also note that the sum of the two independent Poisson streams is Poisson. It can be concluded that the input to every queue is Poisson, so that each queue functions as an Erlang delay system with equilibrium state probabilities given by (4.5) and (4.4) of Chapter 3. Thus, for \( i = 1, 2, 3, 4 \),

\[
P(q_i) = \begin{cases} 
  c_i \frac{(\lambda_i/\mu_i)^{q_i}}{q_i!} & (q_i = 0, \ldots, q_i - 1), \\
  c_i \frac{(\lambda_i/\mu_i)^{q_i}}{q_i!} & (q_i = q_i + 1, \ldots).
\end{cases}
\]

Furthermore, as a consequence of Burke's theorem, the states are independent; that is

\[
P(q_1, q_2, q_3, q_4) = P(q_1)P(q_2)P(q_3)P(q_4).
\]

b With feedback from \( Q_2 \) to \( Q_1 \), the mean arrival rate at \( Q_1 \) is

\[
\lambda_1 = \lambda + (P_2 P_3 + 1 - P_2) \lambda + \cdots
\]

In this particular case, therefore, the mean arrival rate are

\[
\lambda_1 = \frac{\mu_2}{1 - P_2} \lambda, \quad \lambda_2 = \frac{\mu_3}{1 - P_2} \lambda, \quad \lambda_3 = \frac{\mu_4}{1 - P_2} \lambda, \quad \lambda_4 = \lambda.
\]

Let \( \mathbf{p}(q) \) be defined as in (4.12). Then the equilibrium state equations are:

\[
\left( \lambda + \mu_2(q_2) + \mu_3(q_2) + \mu_4(q_2) \right) P(q_1, q_2, q_3, q_4) = \frac{\mu_2(q_2) P(q_1, q_2, q_3, q_4)}{\lambda + \mu_2(q_2) + \mu_3(q_2) + \mu_4(q_2)} \left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \right)
\]

\[
+ \frac{\mu_3(q_2) P(q_1, q_2, q_3, q_4)}{\lambda + \mu_2(q_2) + \mu_3(q_2) + \mu_4(q_2)} \left( \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \right)
\]

\[
+ \frac{\mu_4(q_2) P(q_1, q_2, q_3, q_4)}{\lambda + \mu_2(q_2) + \mu_3(q_2) + \mu_4(q_2)} \left( \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4 \right)
\]

\[
+ \frac{\mu_2(q_2) \mu_3(q_2) P(q_1, q_2, q_3, q_4)}{\lambda + \mu_2(q_2) + \mu_3(q_2) + \mu_4(q_2)} \left( \lambda_1 \lambda_2 \lambda_3 \lambda_4 \right)
\]
Consider the following five equations obtained by pairing terms on LHS and RHS of the equilibrium state equations,

\[ \lambda \mathcal{P}(j_1, j_2, j_3, j_4) = \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4), \]  
\[ \mu_{\lambda}(j_1) \mathcal{P}(j_1, j_2, j_3, j_4) = \lambda \mathcal{P}(j_1, j_2, j_3, j_4), \]  
\[ \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4) = \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4), \]  
\[ \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4) = \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4), \]  
\[ \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4) = \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4). \]

It is easily seen that a solution to (1)-(5) will also be a solution to the equilibrium state equations. By (2), \( \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4) = \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4). \) Substituting into (3) and rewriting lead to

\[ \frac{\lambda}{\mu_{\lambda}(j_3)} \mathcal{P}(j_1, j_2, j_3, j_4) = \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4). \]  

A rewriting of (3) gives

\[ \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4) = \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4), \]

which, by (3), simplifies to

\[ \mu_{\lambda}(j_3) = \frac{\lambda}{\mu_{\lambda}(j_3)} \mathcal{P}(j_1, j_2, j_3, j_4). \]

Hence,

\[ \frac{\lambda}{\mu_{\lambda}(j_3)} \mathcal{P}(j_1, j_2, j_3, j_4) = \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4). \]  

Similarly, by (4) and (5),

\[ \mu_{\lambda}(j_3) = \frac{\lambda}{\mu_{\lambda}(j_3)} \mathcal{P}(j_1, j_2, j_3, j_4), \]

whereby

\[ \frac{\lambda}{\mu_{\lambda}(j_3)} \mathcal{P}(j_1, j_2, j_3, j_4) = \mu_{\lambda}(j_3) \mathcal{P}(j_1, j_2, j_3, j_4). \]
(Chap. H, Ex. 8 b (contd.))

Equations (4), (7), (10), have been derived from and are equivalent to Equations (2), (3), (4). Eq. (7) has the desired form, and we shall keep it the way it is. The final equation, (10), is redundant. To see this, combine (7) and (9) into

\[ \lambda P(Y_1, Y_2, Y_3, Y_4) = M_1(1, 1) P(Y_1, Y_2, Y_3, Y_4) + M_2(1, 1) P(Y_1, Y_2, Y_3, Y_4). \]

By (7) and (10), the right-hand side equals

\[ \left[ \frac{P_{1,2} P_0}{1 - P_0 P_{1,2}} \right] \lambda P(Y_1, Y_2, Y_3, Y_4) = \lambda P(Y_1, Y_2, Y_3, Y_4), \]

since \( P_{1,2} + P_0 = 1 - P_0 P_{1,2} \). This proves redundancy.

We conclude that (7)-(10) are equivalent to the following system of equations,

\[ \lambda_1^* P(Y_1, Y_2, Y_3, Y_4) = M_1(1, 1) P(Y_1, Y_2, Y_3, Y_4), \]
\[ \lambda_2^* P(Y_1, Y_2, Y_3, Y_4) = M_2(1, 1) P(Y_1, Y_2, Y_3, Y_4), \]
\[ \lambda_3^* P(Y_1, Y_2, Y_3, Y_4) = M_3(1, 1) P(Y_1, Y_2, Y_3, Y_4), \]
\[ \lambda_4^* P(Y_1, Y_2, Y_3, Y_4) = M_4(1, 1) P(Y_1, Y_2, Y_3, Y_4). \]

where \( \lambda_i^* \) (\( i = 1, 2, 3, 4 \)) is the mean arrival rate at \( Q_i \).

Recursive solution of Eq. (6), for example, gives, for fixed \( Y_1, Y_2, Y_3, Y_4 \),

\[ P(Y_1, Y_2, Y_3, Y_4) = \begin{cases} P(0, Y_3, Y_4) \frac{\lambda_1^*}{\lambda_i} N_{1,2,3,4} & (Y_i = 0, \ldots, 5 - i), \\ P(0, Y_3, Y_4) \frac{\lambda_2^*}{\lambda_i} N_{1,2,3,4} & (Y_i = 0, 1, \ldots, 4). \end{cases} \]

The marginal probability of \( Y_1, P(Y_1) \), is found by summation over all possible \( Y_2, Y_3, Y_4 \). In general we find

\[ P(Y_1) = \begin{cases} P(0) \frac{\lambda_1^*}{\lambda_i} N_{1,2,3,4} & (Y_1 = 0, \ldots, 4), \\ P(0) \frac{\lambda_2^*}{\lambda_i} N_{1,2,3,4} & (Y_1 = 0, 1, \ldots, 4). \end{cases} \]

In addition it may be shown that the condition for independence holds:

\[ P(Y_1, Y_2, Y_3, Y_4) = P(Y_1) P(Y_2) P(Y_3) P(Y_4) \quad (Y_i = 0, 1, 2, 3, 4) \]
As before, let \( \mu_i(t) = \mu_i \), if \( i = 1, \ldots, n \), \( \mu_i(t) = 0 \), if \( i > n \).

The equilibrium state equations are as follows:

\[
\begin{align*}
\frac{\mu_1(t)}{\mu_2(t)} + \cdots + \frac{\mu_n(t)}{\mu_n(t)} & = M_{n+1}(t) P_{n+1}(t) + \cdots + M_n(t) P_{n+1}(t) + M_1(t) P_1(t) + \cdots + M_n(t) P_n(t) \\
& \quad + \cdots + M_n(t) P_n(t) + \cdots + M_n(t) P_n(t) (j_i = 1, \ldots, m_i, \sum_i = n).
\end{align*}
\]

Consider the following \( m \) equations extracted from the equilibrium state equations,

\[
\begin{align*}
\mu_1(t) P_{11}(t, 1, \ldots, n) & = M_1(t) P_{11}(t, 1, \ldots, n) + 1, \\
\mu_2(t) P_{21}(t, 1, \ldots, n) & = M_2(t) P_{21}(t, 1, \ldots, n), \\
\mu_3(t) P_{31}(t, 1, \ldots, n) & = M_3(t) P_{31}(t, 1, \ldots, n), \\
& \vdots \\
M_m(t) P_{m1}(t, 1, \ldots, n) & = M_m(t) P_{m1}(t, 1, \ldots, n).
\end{align*}
\]

It is clear that a solution to these equations will also be a solution to the equilibrium state equations. Any of the equations can be omitted. We choose to discard (1).

Now write

\[
P_{11}(t, 1, \ldots, n) = \frac{P_{11}(t, 1, \ldots, n)}{P_{11}(t, 1, \ldots, n)} = \frac{P_{11}(t, 1, \ldots, n)}{P_{11}(t, 1, \ldots, n)} \cdots \frac{P_{11}(t, 1, \ldots, n)}{P_{11}(t, 1, \ldots, n)}
\]

By a similar rewriting and the application of Eq. (2)', factor no. 2 on the right can be expressed

\[
\frac{P_{11}(t, 1, \ldots, n)}{P_{11}(t, 1, \ldots, n)} = \frac{P_{11}(t, 1, \ldots, n)}{P_{11}(t, 1, \ldots, n)} \cdots \frac{P_{11}(t, 1, \ldots, n)}{P_{11}(t, 1, \ldots, n)} \cdots \frac{P_{11}(t, 1, \ldots, n)}{P_{11}(t, 1, \ldots, n)}
\]

\[
= \frac{M_1(t)}{M_1(t)} \cdots \frac{M_n(t)}{M_n(t)} \cdots \frac{M_n(t)}{M_n(t)} (j_i = 1, \ldots, n).
\]
(Chap. 4, Ex. 9a)

Proceeding in this manner, using Eq. (k) for factoring factors w.r.t. \( k \), we deduce

\[
P(I,1,2,\ldots,m) = P(n,0,\ldots,0) A_2A_3\cdots A_m,
\]

where, for \( k = 1, 2, \ldots, m, 

\[
A_k = \left\{ \begin{array}{ll}
1 & \text{if } \sum_{i=1}^k i = 0, \\
\frac{\mu_i \left( \sum_{j=1}^k \frac{1}{i} \right)}{\mu_i(i)} - \frac{\mu_i(\sum_{j=1}^k \frac{1}{i})}{\mu_i(i)} & \text{if } \sum_{i=1}^k i \neq 0.
\end{array} \right.
\]

Cancellation of factors in (\*) results in

\[
P(I,1,2,\ldots,m) = \left[ \prod_{i=1}^n \mu_i(n) \right] \frac{\prod Q_i(n)}{\sum_S \prod Q_i(n)} \quad (0 \leq \sum_i n)
\]

where \( Q_i(n) = 1 \) if \( \sum_i n = 0 \), \( Q_i(n) = \left[ \mu_i(n) \mu_i(n-1) \right]^{-1} \) if \( \sum_i n \neq 1 \).

Finally, using \( \sum P(I,1,2,\ldots,m) = 1 \) and substituting the expression for \( \mu_i(n) \), we obtain

\[
P(I,1,2,\ldots,m) = \frac{\prod Q_i(n)}{\sum_S \prod Q_i(n)} \quad (0 \leq \sum_i n)
\]

where \( S = \{ (r_1,r_2,\ldots,r_m) : 0 \leq r_1 \leq r_2 \leq \cdots \leq r_m \leq n \} \) and

\[
Q_i(n) = \left\{ \begin{array}{ll}
\frac{(1/\mu_i)^{r_1}}{n^{r_1}} & \text{if } r_1 < n, \\
\frac{(1/\mu_i)^{r_1}}{n^{r_1}} & \text{if } r_1 = n, \\
(1/\mu_i^{r_1}) & \text{if } r_1 > n.
\end{array} \right.
\]

The probability \( P(I,1,2,\ldots,m) \) has been written as a product of factors dependent on \( I,1,2,\ldots,m \) respectively, but the random variables \( N_1, N_2, \ldots, N_m \) are not independent. The reason is that the set \( S \) for which Eq. (1) applies is not a product space \( I_1 \times I_2 \times \ldots \times I_m \). If, for instance, \( N_1 = n \), then \( N_2 = 0 \), so that, obviously, \( N_1 \) and \( N_2 \) cannot be independent variables.
In the general model, where a departure from $Q_i$ with probability $p_{ij}$ goes to $Q_j$ ($i = 1, 2, ..., m$), the equilibrium state equations are

$$
\sum_{x_{i+1}} M(x_{i+1}) P(x_{i+1}, ..., x_m) = \sum_{x_{i+1}} \sum_{k=1}^{n} M(x_{i+1}, k) p_{ik} P(x_{i+1}, r_i, ..., r_m) + \sum_{x_{i+1}} M(x_{i+1}) P(x_{i+1}, ..., x_m).
$$

We shall verify that the solution is of the form (1), that is $P(x_{i+1}, ..., x_m) = \prod_{i+1} Q_i(x_i) / \sum_{i+1} \prod_{i+1} Q_i(x_i)$, but where, for some $x_i$,

$$
Q_i(x_i) = \begin{cases} (1/\lambda_i)^k & (j_i < x_i), \\ (1/\lambda_i)^{x_i} & (j_i = x_i), \\ (1/\lambda_i)^{x_i} \prod_{j=1}^{i-1} (1/\lambda_j)^{x_j} & (j_i > x_i). \end{cases}
$$

Substitution of (1) into (2) gives

$$
\left( \prod_{i+1} Q_i(x_i) \right) \sum_{x_{i+1}} M(x_{i+1}) = \left( \prod_{i+1} Q_i(x_i) \right) \sum_{x_{i+1}} \sum_{k=1}^{n} M(x_{i+1}, k) p_{ik} Q_i(x_i - 1) Q_j(x_i) / \prod_{i+1} Q_i(x_i) + \left( \prod_{i+1} Q_i(x_i) \right) \sum_{x_{i+1}} M(x_{i+1}) p_{ik}.
$$

Cancellation of the common factor $\prod_{i+1} Q_i(x_i)$ and the use of (3) results in

$$
\sum_{x_{i+1}} M(x_{i+1}) = \sum_{x_{i+1}} \sum_{k=1}^{n} M(x_{i+1}, k) p_{ik} Q_i(x_i - 1) Q_j(x_i) / \prod_{i+1} Q_i(x_i) + \sum_{x_{i+1}} M(x_{i+1}) p_{ik}.
$$

Hence,

$$
\sum_{x_{i+1}} M(x_{i+1}) \left[ 1 - \lambda_i x_i \sum_{k=1}^{n} M(x_{i+1}, k) p_{ik} / \lambda_i - p_{ik} \right] = 0,
$$

whereby

$$
\sum_{x_{i+1}} M(x_{i+1}) \left[ 1 - \lambda_i x_i \sum_{k=1}^{n} M(x_{i+1}, k) p_{ik} / \lambda_i \right] = 0.
$$

This leads to the requirement that

$$
\sum_{x_{i+1}} p_{ik} (M(x_{i+1}, k)) = \lambda_i x_i (i = 1, 2, ..., m).
$$

A solution $\{\lambda_i, x_i\}$ exists provided all roots "communicate."
The following model can be used…

OBS! The server groups have been renamed: G₁ ⇀ H₁, G₂ ⇀ H₂.

For group H₁ (digit trunks for dialing) let s₁ be the number of (exponential) servers with service rate μ₁, and for group H₂ (time slots for talking) let s₂ be the number of (exponential) servers with service rate μ₂. The possibility that all servers in H₁ are busy, while a server in H₂ is idle, implies s₁ < s₂, since a call holding a server in H₁ will at the same time hold a server in H₂. Let I₁ = calls in dialing phase or in waiting position, and let I₂ = calls in talking phase. Let \( \lambda I₁ \neq \mu I₁ \) if \( s₁ < s₂ \).

The equilibrium state equations are, for \( j₁, j₂ \geq 0 \),

\[
(\lambda + \mu(j₁) + 2 \mu(j₂)) P(j₁,j₂) = \begin{cases} 
(0 \leq j₁, j₂ < s₂) \\
\lambda P(j₁-1,j₂) + \mu(j₁)P(j₁,j₂-1) + (j₁+2\mu)P(j₁,j₂+1), \\
\lambda P(j₁-1,j₂) + \mu(j₁)P(j₁,j₂-1). 
\end{cases}
\]

Hence the equations

\[
\begin{align*}
\lambda P(j₁,j₂) &= (j₁+2\mu)P(j₁,j₂+1) \quad (0 \leq j₁, j₂ < s₂), \\
\mu(j₁)P(j₁,j₂) &= \lambda P(j₁-1,j₂) \quad (0 \leq j₁, j₂ < s₂), \\
\mu(j₂)P(j₁,j₂) &= \lambda P(j₁,j₂-1) \quad (0 \leq j₁, j₂ < s₂),
\end{align*}
\]

a solution to which will also solve the equilibrium state equations. Disregarding the last, redundant equation, solving the other two recursively, starting with P(0,0), we find

\[
P(j₁,j₂) = c \cdot P(j₁) P(j₂) \quad (0 \leq j₁, j₂ < s₂),
\]

where

\[
\begin{align*}
P(j₁) &= \left( \frac{\lambda}{\mu(j₁)} \right)^{j₁} P(0,j₁) \quad (j₁ \leq s₁), \\
P(j₂) &= \left( \frac{\lambda}{\mu(j₂)} \right)^{j₂} P(0,j₂) \quad (j₂ \leq s₂), \\
P(j₁,j₂) &= \frac{\lambda^{j₁} \mu^{j₂}}{\lambda^{j₁} + \mu^{j₂}} \quad (j₁ > s₁, j₂ > s₂).
\end{align*}
\]

The constant c is determined by \( \sum_{j₁,j₂} P(j₁,j₂) = 1 \).
The following is a simplified version of a model...

In the present model, the customers are of any of three types, going through phases whose duration has an exponential distribution. The overall arrival rate is \( \lambda \), and with probability \( p_r \) \((r=1,2,3)\) a fire is of type \( r \). A fire of type 1 is characterized by a single phase with mean \( \tau(1) \), requiring the use of 1 fire engine. A fire of type 2 goes through three phases with means \( \tau(1), \tau(2), \tau(3) \), requiring 1, 2, and 1 fire engines, respectively. A fire of type 3 has two phases with means \( \tau(1) \) and \( \tau(3) \), requiring 2 and 1 fire engines, respectively.

The fires pass through a network of infinite-server queues. In the figure, the variables \( j_1, j_2, j_3, k_1, k_2 \) denote the number of fires in progress, and the numbers in square brackets indicate how many fire engines are needed in each phase.

In reality, the network is composed of three independent queuing systems with arrival rates \( \lambda_1, \lambda_2, \lambda_3 \), respectively. The first queuing system is an infinite-server queue, whose equilibrium distribution, by Eq. (4.27) of Chapter 2, is \( F = \left(1 - \frac{\lambda \tau_1}{\mu_1}\right) e^{-\lambda \tau_1} \). The other two queuing systems are tandem queues. By Burke's theorem, the input to each constituent queue is Poisson so that also here the state variables follow a Poisson distribution, and furthermore, the equilibrium states of the queues are independent. Hence,

\[
P(j_1, j_2, j_3, k_1, k_2) = \frac{e^{-\lambda \tau_1}}{j_1!} \frac{e^{-\lambda_2 \tau_2}}{j_2!} \frac{e^{-\lambda_3 \tau_3}}{j_3!} \frac{e^{-\lambda_2 \tau_2}}{k_1!} \frac{e^{-\lambda_3 \tau_3}}{k_2!} \ c,\]

where,

\[
c = \exp \left[ -\frac{\lambda \tau_1}{\mu_1} - \frac{\lambda_2 \tau_2}{\mu_2} - \frac{\lambda_3 \tau_3}{\mu_3} - \frac{\lambda_2 \tau_2}{k_1} - \frac{\lambda_3 \tau_3}{k_2} \right].\]

The distribution of \( m = j_1 + j_2 + j_3 + k_1 + k_2 \) is found by convoluting \( j_1, j_2, j_3, k_1, k_2 \).
Chapter 4, Exercise 12

(a) The equilibrium state equations for the distribution \( h(j, k_1, k_2) \) are

\[
\begin{align*}
(1 + \lambda + \mu_1 + \mu_2) h(j, k_1, k_2) &= a h(j-1, k_1, k_2) \\
(\mu_1 + \lambda + \mu_2) h(j, k_1, k_2) &= \mu_2 h(j+1, k_1, k_2) + \mu_1 h(j, k_1-1, k_2) + \mu_1 h(j, k_1, k_2-1)
\end{align*}
\]

\( j = 0 \)

\[
\begin{align*}
(1 + \lambda + \mu_1 + \mu_2) h(j, k_1, k_2) &= a h(j-1, k_1, k_2) \\
(\mu_1 + \lambda + \mu_2) h(j, k_1, k_2) &= \mu_2 h(j+1, k_1, k_2) + \mu_1 h(j, k_1-1, k_2) + \mu_1 h(j, k_1, k_2-1)
\end{align*}
\]

\( j = 0 \)

We shall verify that

\[
h(j, k_1, k_2) = P(j, k) \binom{k_1}{j} \left( \frac{\lambda}{\mu_1} \right)^j \left( \frac{\mu_1}{\mu_1} \right)^{k_1-j}
\]

where \( k = k_1 + k_2 \). We begin by showing that the suggested solution satisfies the equilibrium state equations. Insertion of the above expression for \( h(j, k_1, k_2) \) into the two sets of equilibrium state equations, and a straightforward reduction, result in Eqs. (35) and (36), respectively, which are always satisfied. Hence, the suggested solution is indeed the solution, at least up to a factor. Now, it is easily shown that \( \Sigma_{k_1+k_2=N} h(j, k_1, k_2) = P(j, k) \) by the given expression, just as it should, as the expression does give the correct value.

(b) By definition of a conditional probability, \( h(j, k_1, k_2) \) may be expressed as a function of \( h(j, k_1, k_2) = P(j, k) P(k_1+k_2, N-k | N=k) \). By comparison with (4), therefore

\[
P_N(k_1, k_2 | N=k) = \binom{k}{k_1} \left( \frac{\lambda}{\mu_1} \right)^{k_1} \left( \frac{\mu_1}{\mu_1} \right)^{k_2}
\]

(c) A binomial variable \( X \) with parameters \( (n, p) \) has mean \( E(X) = np \) and variance \( \text{Var}(X) = np(1-p) \). Hence, \( E(X^2) = \text{Var}(X) + E(X) = np(1-p) + np \). By (d), for \( N=k \), \( N+k \) is a binomial variable with \( (n, p) = (k, 1/p) \). It follows that

\[
E(N_k | N=k) = k \frac{\lambda}{\mu_1}
\]

\[
E(N_k^2 | N=k) = k(k-1) \left( \frac{\lambda}{\mu_1} \right)^2 + k \frac{\mu_1}{\mu_1}
\]
(Chap. 4, Ex. 12 a)

\[ \Box \]

By (5), \( E(N) = E(N; N=K) - E(N; N=0) \). Hence,
\[ E(N) = \frac{a}{\lambda} E(N). \]  \( \text{(i)} \)

By (6), \( E(N^2) = E(N; N=N+K) = E(N; N=(N-1)(\frac{a}{\lambda}) + N/\lambda) \). Hence,
\[ E(N^2) = \left( \frac{a}{\lambda} \right)^2 E(N) + \frac{\lambda}{\lambda} \left( 1 - \frac{a}{\lambda} \right) E(N). \]  \( \text{(ii)} \)

By (1) and (7), \( V(N) = E(N) - E^2(N) = \left( \frac{a}{\lambda} \right)^2 \lbrack E(N^2) - E(N) \rbrack + \frac{\lambda}{\lambda} \left( 1 - \frac{a}{\lambda} \right) E(N). \). Hence,
\[ V(N) = \left( \frac{a}{\lambda} \right)^2 V(N) + \frac{\lambda}{\lambda} \left( 1 - \frac{a}{\lambda} \right) E(N). \]  \( \text{(2)} \)

Dividing Eq. (i) into Eq. (2), we get
\[ \frac{V(N)}{E(N)} = \frac{\lambda}{\lambda} \frac{V(N)}{E(N)} + \left( 1 - \frac{a}{\lambda} \right). \]

Letting \( \alpha = V(N)/E(N) \) and \( \rho = V(N)/E(N) \), we find, for \( \rho_1 = a/\lambda \),
\[ \rho_1 - 1 = \rho_1 (\rho_1 - 1). \]  \( \text{(3)} \)

By (5) and (6), when \( n = 2 \),
\[ E(N_2) = E(N; N=K) - E(N; N=2) \]
\[ = \lambda E(N; N=K) - E(N; N=0) \]
\[ = \lambda \left( \frac{a}{\lambda} \right)^2 - \left( \frac{a}{\lambda} \right) \frac{\lambda}{\lambda} \left( 1 - \frac{a}{\lambda} \right) \]
\[ = \left( \frac{a}{\lambda} \right)^2 \left( 1 - \frac{a}{\lambda} \right). \]

Hence, \( E(N_2) = \frac{a}{\lambda} E(N; N=K) = \frac{a}{\lambda} \left( 1 - \frac{a}{\lambda} \right) E(N). \), or
\[ E(N; N_2) = \frac{a}{\lambda} \frac{\lambda}{\lambda} \left[ E(N^2) - E(N) \right]. \]

By (1),
\[ E(N)E(N_2) = \frac{a}{\lambda} \frac{\lambda}{\lambda} E(N). \]

Thus,
\[ Cov(N_1, N_2) = E(N_1N_2) - E(N_1)E(N_2) = \frac{a}{\lambda} \frac{\lambda}{\lambda} \left[ E(N^2) - E(N) \right]. \]  \( \Box \)
Chapter 4, Exercise 13

'Show that when \( a_1 = a_2 = \cdots = a_{k-1} = 0 \) and \( a_k = a \), then ...

When \( \lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0 \) and \( \lambda_k = \lambda \), then Eq. (4.6b) becomes

\[
\lambda_0 P(j-l) = j \mu P(j) \quad (j = 1, 2, \ldots, s-1),
\]

\[
\lambda_0 P(s-l) + \lambda_s P(s) = \mu P(s) \quad (j = s),
\]

or,

\[
aP(j-l) = \mu P(j) \quad (j = 1, 2, \ldots, s-1),
\]

\[
aP(s-l) = (s-a)P(s) \quad (j = s).
\]

The solution, in terms of \( P(0) \), is

\[
P(j) = \frac{\mu^{j-l}}{j!} P(0) \quad (j = 0, 1, \ldots, s-1),
\]

\[
P(s) = \frac{a^s}{s! (1-a/\mu)} P(0).
\]

Hence, and by Eq. (4.8) of Chapter 3, if \( a < \mu \),

\[
P(s) = \frac{a^s}{s! (1-a/\mu)} \frac{a^s}{s! (1-a/\mu)} = C(s, \mu).
\]

Chapter 4, Exercise 14

'Consider again the premise of Exercise 7 ...

For a BCD queue with heterogeneous, exponential servers and random selection of server, let \( P(x_1, x_2, k) = P(x_1 x_2 x_3, x_k, k = k) \).

When \( \sum x_i < \alpha \), the equilibrium state equations are precisely as in the B/C/B queue of Exercise 7, with \( P(x_1, x_2, 0) \) replacing \( P(x_1, x_2) \).

When \( \sum x_i = \alpha \) (\( x_i = 1 \) for all \( i \)) and \( k = 0 \), we now have

\[
(\lambda + \sum x_i \mu_i) P(x_1, x_2, 0) = \left( \frac{\lambda}{\mu_i} x_i + s \right)
\]

\[
(\sum \mu_i) P(x_1, x_2, 1)
\]

\[
+ \lambda (P(x_{l-1}, x_{l-2}, x_{l-1}, 0) + P(x_{l-1}, x_{l-2}, x_{l-1}, 0) + \cdots + P(x_{l-1}, x_{l-2}, x_{l-1}, 0)).
\]
(Chap. 4, Ex. 14)

In addition we have the rate up - rate down equations

\[ \lambda P(l_{,1}, l, k) = (\sum_{i=1}^{k} \mu_i) P(l_{,1}, l, k+1) \quad (k \geq 0, l). \quad (\star \star) \]

By subtraction of Eq. (\star \star \star), for \( k = 0 \), from Eq. (\star \star), we derive

\[ (\sum_{i=1}^{l} \mu_i) P(x_{,1}, x_{,0}, 0) = (\sum_{i=1}^{l} \mu_i) x_{,0} \quad (\star \star \star) \]

\[ \lambda P(y_{,1}, x_{,0}, 0) = \lambda P(x_{,1}, x_{,0}, 0) - \lambda P(x_{,1}, x_{,0}, 0) - \lambda P(x_{,1}, x_{,0}, 0). \]

Observe that (\star \star \star) is the remaining equilibrium state equation in the BCC case of Exercise 7, with \( P(x_{,1}, x_{,0}, 0) \) instead of \( P(y_{,1}, x_{,0}, 0) \).

We conclude that for \( k = 0 \) the solution is the same as in the BCC case except for a proportionality constant,

\[ P(x_{,1}, x_{,0}, 0) = c \bar{P}(x_{,1}, x_{,0}) \quad (x_{,0} > 0, l \geq 1, n). \quad (1) \]

By (\star \star \star),

\[ P(l_{,1}, l, k) = (\sum_{i=1}^{l} \mu_i) P(l_{,1}, l, k) \quad (k \geq 1, l, n). \]

Utilization of (1) and the definition \( \rho = \lambda / \sum_{i=1}^{l} \mu_i \) (utilization factor) give

\[ P(l_{,1}, l, k) = c \rho^k \bar{P}(l_{,1}, l) \quad (k \geq 1, l, n). \quad (2) \]

Substitution into the normalization equation leads to

\[ 1 = \sum_{x_{,0}} P(x_{,1}, x_{,0}, 0) + \sum_{k=1}^{\infty} P(l_{,1}, l, k) \]

\[ = c \sum_{x_{,0}} \bar{P}(x_{,1}, x_{,0}) + c \sum_{l=1}^{\infty} \rho^l \bar{P}(l_{,1}, l) \]

\[ = c + c \bar{P}(l_{,1}, l) \sum_{l=1}^{\infty} \rho^l \]

Hence,

\[ c = (1 - \frac{\rho}{1-\rho} \bar{P}(l_{,1}, l))^{-1}. \quad (3) \]

Thus, the equilibrium probabilities for the BCC queue are given by (1), (2), and (3), whereas \( \{ P(x_{,1}, x_{,0}, 0) \} \) are the equilibrium probabilities of the corresponding BCC queue of Exercise 7.
Chapter 4, Exercise 15

Show that if \( a > 0 \), then \( \psi = \alpha \) when \( \nu = 0 \), \( \psi > \alpha \) when \( \nu > 1 \).

\[
\alpha = \frac{\lambda B(\lambda, \alpha)}{\alpha + \lambda B(\lambda, \alpha)}.
\]

(3.1)

\[
\psi = \alpha (1 - \frac{\alpha}{\alpha + 1 + \alpha - \alpha}).
\]

(3.2)

\( \alpha = 0 \). Clearly, \( B(\lambda, \alpha) = 1 \). By (3.1) and (3.2) then, \( \psi = \alpha = 0 \).

This should come as no surprise, as the equilibrium state of an infinite-server system with Poisson input has the Poisson distribution with parameter (mean and variance) equal to \( \lambda \).

\( \delta \leq 1 \). By Exercise 6 of Chapter 3,

\[
B(\delta, \alpha) = \frac{\lambda B(\lambda, \alpha)}{\alpha + \lambda B(\lambda, \alpha)}.
\]

(3.1)

Hence, follow the two equivalent equations

\[
\frac{\delta + 1 + \lambda B(\lambda, \alpha)}{\delta + 1 + \lambda B(\lambda, \alpha)} = \alpha B(\lambda, \alpha)
\]

(3.2.0), (\( \delta \geq 0 \)).

\[
B(\delta + 1, \alpha) = \frac{\alpha B(\lambda, \alpha)}{\alpha B(\lambda, \alpha) - \lambda B(\lambda, \alpha)}
\]

(3.2.0). (\( \delta 

Now, for \( \delta \leq 1 \),

\[
\psi > \alpha \Leftrightarrow \frac{\alpha}{\alpha + 1 + \alpha - \alpha} > \alpha
\]

(by 3.1)

\[
\psi > \alpha \Leftrightarrow \frac{\alpha}{\alpha + 1 + \alpha - \alpha} > \alpha \Rightarrow \alpha \lambda B(\lambda, \alpha)
\]

(by 3.1)

\[
\psi > \alpha \Leftrightarrow \frac{\alpha B(\lambda, \alpha)}{\alpha B(\lambda, \alpha) - \lambda B(\lambda, \alpha)} > \alpha\lambda B(\lambda, \alpha) - \lambda B(\lambda, \alpha)
\]

(by 3.1)

\[
\psi > \alpha \Leftrightarrow \frac{\alpha B(\lambda, \alpha)}{\alpha B(\lambda, \alpha) - \lambda B(\lambda, \alpha)} > \alpha\lambda B(\lambda, \alpha) - \lambda B(\lambda, \alpha)
\]

(by 3.1)

As usual, let \( \mu_i \) denote the load on the \( i \)th service in an Erlang loss system. By Eq. (5.15) of Chapter 3, \( \mu_i = \lambda B(\lambda, \alpha) - \lambda B(\lambda, \alpha) \) and

\[
\Sigma_i \mu_i = \alpha[1 - B(\lambda, \lambda)]
\]

(\( \delta \geq 0 \)). Thus, for \( \delta \leq 1 \),

\[
\psi > \alpha \Rightarrow \frac{\psi}{\psi + \lambda B(\lambda, \alpha)} > \frac{\delta}{\delta + 1}
\]

(by 3.1)

According to Messerli (1971), \( \psi \geq \psi > \psi > \ldots \).

Hence, \( \delta \leq 1 \Rightarrow \psi > \alpha \).

\( \square \)
Chapter 4, Exercise 16

"o. a. 10 Erl of Poisson traffic is offered to a group of 10 servers."

The exercise requires the use of the Equivalent Random Method. The case is described by Figure 4-8 for \( n = 2 \). Offered loads and server group size & are:

Parts (a) and (b): \((\lambda, \mu) = (10,10), (\lambda, \mu) = (5,5)\), \( c \) to be calculated.
Part (c):
\((\lambda, \mu) = (10,15), (\lambda, \mu) = (5,7.5)\), \( c \) as in (a) & (b).

The values of \( B(\lambda, \mu) \) and \( B(\mu, \lambda) \) may be read off Figure A1 in the Appendix. Exact values obtained from tables of the Binom distribution or the Erlang B formula are:

\[
\begin{align*}
B(\lambda, \mu) & = B(10,10) = 0.215, \\
B(\mu, \lambda) & = B(5,5) = 0.285.
\end{align*}
\]

For the two primary groups, the mean and the variance of the equilibrium state of an infinite-server backup group are, by Eqs. (3.1) and (3.2),

\[
\begin{align*}
\nu_1 & = 2.15, \\
\nu_2 & = 4.35, \\
\nu_3 & = 1.42, \\
\nu_4 & = 2.34.
\end{align*}
\]

Hence, the total overflow is characterized by the parameters

\[
\begin{align*}
\lambda & = \nu_1 + \nu_2 = 3.57, \\
\sigma & = \nu_3 + \nu_4 = 6.69, \\
S & = \sigma/\lambda = 1.87
\end{align*}
\]

whereby

\[
\tilde{\alpha} = \sigma + 3\alpha(1-\alpha) = 11.6 \quad [\text{by (7.15)}],
\]

\[
\alpha = \frac{\tilde{\alpha}}{\alpha+1} \approx 9 \quad [\text{by (7.16)}].
\]

Hence,

\[
\begin{align*}
\sigma & = \sqrt{\frac{3}{8}} = 9, \\
\alpha & = \sqrt{\frac{3}{8} + 2\alpha - \alpha^2} = 11.1 \quad [\text{by (7.17)}].
\end{align*}
\]

Thus, \((\lambda, \mu) = (9, 11.1)\) define the equivalent random system having the approximate overflow characteristics \( \alpha \) and \( \sigma \) of the total overflow from the two primary groups.
(Chap 4, Ex. 16 a.)

When the overflow group has size \( c \), the loss among the overflow customers (from the equivalent random group) is, by (7.13),

\[
\tau_c = \frac{B(s+c,a)}{B(s,a)} = \frac{B(9-c,11,1)}{B(9,11,1)}.
\]

By Figure A-1, \( B(9,11,1) = 0.333 \). Thus the requirement

\[
\tau_c < 0.18
\]

translates into the condition

\[
B(9-c,11,1) < 0.0323 < B(9+c-1,11,1).
\]

From Figure A-1 we obtain

\[
B(17,11,1) = 0.0259, \quad B(6,11,1) = 0.0408.
\]

Hence, our estimate of the necessary capacity, i.e. size, of the overflow group becomes

\[
c = 17 - 5 = 12,
\]

(1)

corresponding to an estimated loss on the overflow customers equal to

\[
\tau_c = \frac{B(17,11,1)}{B(9,11,1)} = \frac{0.0259}{0.333} = 0.077.
\]

(2)

b) The estimate of the loss for the system as a whole is, by (7.14),

\[
\tau = \frac{2B(s+c,a)}{a} = \frac{11,11B(17,11,1)}{16*5} = \frac{0.237}{15} = 0.019
\]

(3)

c) After increasing the load on the primary group by 50%, the new offered loads will be \( a^* = 15 \) and \( a^*_1 = 7.5 \). We shall estimate the losses on the overflow group as well as the system as a whole, assuming the old primary group sizes, \( a_1 = 10 \), \( a_2 = 5 \), and \( c = 12 \). To begin, the losses on the primary groups are, by Fig A-1,

\[
B(s_1,a^*_1) = B(10,15) = 0.210,
\]

\[
B(s_2,a^*_2) = B(5,7.5) = 0.453.
\]
(Chap. 4, Ex. 16 a)

The calculation of an approximate equivalent random system proceeds along the same lines as in part (a). The results are as follows,

\[ x^* = 6.15, \quad y^* = 11.23, \]
\[ a^* = 3.30, \quad x^*_1 = 5.26, \]
\[ x^* = x^*_2 + x^*_3 = 9.55, \]
\[ y^* = y^*_1 + y^*_2 = 16.49, \]
\[ s^* = \sqrt{y^*} = 1.73, \]
\[ \tilde{a}^* = y^* + 3(x^* - 1) = 20.3, \]
\[ \tilde{s}^* = \frac{1}{(x^*_2 + x^*_3) - a^* - 1} = 0.97, \]
\[ s^* = \frac{\tilde{a}^*}{\tilde{s}^*} = 11, \]
\[ a^* = \frac{1}{s^* + 1} = 19.6. \]

Thus, \((x^*, y^*) = (11.19, 4.6)\) define the equivalent random system having the approximate overflow characteristics \(x^*\) and \(y^*\) of the total overflow from the two primary groups, after the 50% interest in loads.

The estimated loss on the overflow group is now calculated to, by the use of Figure A-2 in the Appendix,

\[ \Pi^* = \frac{\beta_a(x^*_2, y^*)}{\beta_a(x^*, y^*)} = \frac{0.129}{0.199} \]
\[ = 0.649, \]

and the estimated loss for the system as a whole is

\[ \Pi^* = \frac{\alpha^* \beta_a(x^*_2, y^*)}{\alpha^*_1 \beta_a(x^*, y^*)} = \frac{0.129 \times 19.46}{15.39} \]
\[ = 3.51 \]
\[ = 0.156. \]
Chapter 4, Exercise 19

Poisson traffic totaling 10 art is offered to a group of 10 servers.

\[ a_1 = \lambda_1 = 5 \quad s_0 = 10 \quad a_2 = \lambda_2 = 5 \]

Total overflow

As in a similar case in Exercise 16, the total overflow from the primary group is characterized by the mean and the variance of the equilibrium state of a fictitious infinite-server back-up group, equal to

\[ \alpha = (a_1 + a_2) B(s_0, a_1 + a_2) = 2.15, \]

\[ \sigma = \alpha (1 - \alpha + \frac{a_1 + a_2}{s_0 + a_1 + a_2 (a_1 + a_2)}) = 4.35. \]

High-priority overflow

By Exercise 12, mean and variance of the high-priority overflow stream are, respectively,

\[ \alpha_1 = \frac{a_1}{a_1 + a_2} \alpha = a_1 B(s_0, a_1 + a_2) = 1.07, \]

\[ \sigma_1 = (\frac{a_1}{a_1 + a_2})^2 \sigma + \frac{a_1}{a_1 + a_2} (1 - \frac{a_1}{a_1 + a_2}) \alpha = 1.62. \]

Equivalent random system

The decision on \( c \) will be based upon a calculation of an equivalent random system \((s,\alpha)\) whose overflow has approximately the mean and the variance of the overflow stream of high-priority customers from the primary group. First, we calculate

\[ \varepsilon = \frac{\sigma_1}{\alpha_1} = 1.51. \]
As a first approximation we calculate
\[ \bar{a} = \bar{a} + 3 \bar{a} (\bar{a} - 1) = 3.93 \quad \text{[by (7.15)]}, \]
\[ \bar{s} = \frac{\bar{a} + \bar{s}}{\bar{a} + \bar{s} - 1} - \bar{a} - 1 = 4.35 \quad \text{[by (7.16)]}. \]

Hence, by (7.17),
\[ \mu \approx \frac{\mu (\bar{a} + \bar{s})}{\bar{a} + \bar{s}} = 3.72, \]
so that the equivalent random system is described by
\[ (\bar{a}, \bar{s}) = (4, 3.72). \]

**Calculation of \( c \)**

Our estimate of the loss of high-priority customers in the system as a whole is:
\[ \Pi = \frac{2B(\bar{a} + \bar{s}, \bar{s})}{\bar{a}} = \frac{3.72 B(4+c, 3.72)}{\bar{s}}. \]

The smallest \( c \) meeting the requirement \( \Pi < 0.01 \) therefore must satisfy the inequalities
\[ B(4+c, 3.72) < 0.0124 < B(4+c-1, 3.72) \]

By Figure A-1,
\[ B(4, 3.72) = 0.0092, \quad B(3, 3.72) = 0.0023. \]

It follows that the size of the overflow group should be
\[ c = 4 - \bar{s} = 5, \]
for which
\[ \Pi = 0.0068. \]
Consider the Erlang loss system with hyperexponential service times.

Let $j_1$ be the number of customers whose service time is exponential with mean $\mu_1$ (with probability $p_1$), and let $j_2$ be the number of customers whose service time is exponential with mean $\mu_2$ (with probability $p_2$). Let $P(j_1, j_2)$ denote the equilibrium probability that the state of the system is $(j_1, j_2)$. The conservation-of-flow equations are:

\[
\begin{align*}
&\lambda_1 p_1 P(j_1+1, j_2) + \lambda_2 p_2 P(j_1, j_2+1) = (j_1 + 1)\mu_1 P(j_1, j_2) \quad (0 \leq j_1, j_2 < s) \\
&\lambda_1 p_1 P(j_1-1, j_2) + \lambda_2 p_2 P(j_1, j_2-1) + (j_1 + j_2)\mu_2 P(j_1, j_2) = (j_1 + j_2)\mu_2 P(j_1, j_2) \quad (j_1, j_2 \geq s).
\end{align*}
\]

From these equations, we extract the following two sets of equations,

\[
\begin{align*}
&\lambda_1 p_1 P(j_1+1, j_2) = (j_1 + 1)\mu_1 P(j_1, j_2), \quad (0 \leq j_1 < s), \\
&\lambda_2 p_2 P(j_1, j_2+1) = (j_2 + 1)\mu_2 P(j_1, j_2), \quad (0 \leq j_2 < s).
\end{align*}
\]

The solution of the above equations, which also is a solution of the equilibrium state equations, is

\[
P(j_1, j_2) = \frac{(\lambda_1 p_1) j_1}{j_1!} \frac{(\lambda_2 p_2) j_2}{j_2!} c \quad (0 \leq j_1, j_2 \leq s).
\]

It follows that the equilibrium probability that altogether $j$ customers will be in service equals

\[
P_j = \sum_{j_1+j_2=j} P(j_1, j_2) = \frac{(\lambda_1 p_1) j^2}{j!} \frac{(\lambda_2 p_2) j^2}{j!} c \quad (0 \leq j \leq s).
\]

Introducing the unconditional mean service time by

\[
\frac{1}{\mu} = \frac{P_j}{P_1} + \frac{P_j}{P_1}.
\]

it is readily verified that

\[
P_j = \sum_{k=0}^{\infty} \frac{(\lambda/k!)}{\sum_{k=0}^{\infty} (\lambda/k!)/k!} \quad (0 \leq j \leq s).
Show that if a random variable $X$...

We need to show that if $X_i$ is an exponentially distributed variable with parameter $\mu_i$, then

$E(X_i) = \frac{\mu_i}{\lambda_i}$,

$V(X_i) = \frac{\mu_i^2}{\lambda_i^2}$,

$V(X) = E(X^2) - E^2(X) = \mu_i^2$.

**Case 1:** $X$ is a sum of independent, exponentially distributed variables.

Clearly,

$E(X) = \sum_i E(X_i) = \sum_i \mu_i$,

$V(X) = \sum_i V(X_i) = \sum_i \mu_i^2$.

Hence,

$E(X) = \left(\sum_i \mu_i^2\right)^2 = \sum_i \mu_i + \sum_i \mu_i^2 = \sum_i \mu_i^2 = V(X)$.

Since $E(X) > 0$, we can conclude that

$E(X) > \sqrt{V(X)}$. (1)

**Case 2:** $X$ is a mixture of independent, exponentially distributed variables.

Clearly,

$E(X) = \sum \pi_i E(X_i) = \sum \pi_i \mu_i$,

$V(X) = \sum \pi_i V(X_i) = 2 \sum \pi_i \mu_i^2$.

By Schwarz inequality, $\left(\sum \pi_i \mu_i^2\right) \geq \sum \pi_i \mu_i^2 \sum \pi_i \mu_i^2$. Hence,

$\left(\sum \pi_i \mu_i^2\right)^2 = \sum \pi_i \mu_i^2 \sum \pi_i \mu_i^2$,

whereby the inequality

$V(X) = E(X^2) - E^2(X) = 2 \sum \pi_i \mu_i^2 - \left(\sum \pi_i \mu_i^2\right)^2 > 0$.

Thus, in this case, $E(X) < \sqrt{V(X)}$. (2)
Chapter 5, Exercise 1

Consider a birth-and-death process.

(a) Consider the Markov chain of states immediately following events, where an event is either an arrival (causing a change of state) or a departure. Denote by $E_i$, $E_{i+1}$, $E_{i+2}$, three such consecutive states in statistical equilibrium. For each $j \geq 0$, by the Markov property,

$$P(E_i^{(*)} \rightarrow E_{i+1}, E_{i+1}^{(*)} \rightarrow E_j) = P(E_i^{(*)} \rightarrow E_j, E_i^{(*)} \rightarrow E_{i+1}, E_{i+1} \rightarrow E_j)$$

$$+ P(E_i^{(*)} \rightarrow E_{i+1}, E_i^{(*)} \rightarrow E_{i+1}, E_{i+1} \rightarrow E_j)$$

$$= P(E_i^{(*)} \rightarrow E_{i+1}, E_{i+1} \rightarrow E_j) P(E_{i+1} \rightarrow E_j | E_i^{(*)} \rightarrow E_{i+1})$$

$$+ P(E_i^{(*)} \rightarrow E_{i+1}, E_i^{(*)} \rightarrow E_{i+1}, E_{i+1} \rightarrow E_j) P(E_{i+1} \rightarrow E_j | E_i^{(*)} \rightarrow E_{i+1}).$$

Now,

$$P(E_i^{(*)} \rightarrow E_i, E_{i+1} \rightarrow E_j) = \frac{1}{2} \Pi_i^{(*)},$$

$$P(E_i^{(*)} \rightarrow E_j, E_{i+1} \rightarrow E_{i+1}) = \frac{1}{2} \Pi_j,$$

$$P(E_i^{(*)} \rightarrow E_{i+1}, E_{i+1} \rightarrow E_{i+1}) = \frac{1}{2} \Pi_{i+1}^{(*)}.$$

The first equation, for instance, holds because with probability \( \frac{1}{2} \), an event is a departure, and the conditional probability of departure state $E_j$ given a departure equals \( \Pi_j \). Inserting these expressions and writing $P(E_i^{(*)} \rightarrow E_j | E_i^{(*)} \rightarrow E_{i+1}) = P(E_{i+1} \rightarrow E_j)$ we obtain, for each $j \geq 0$,

$$\Pi_j^{(*)} = \frac{\Pi_j}{\Pi_j} P(E_{i+1} \rightarrow E_j) + \frac{\Pi_{i+1}^{(*)}}{\Pi_i^{(*)}} P(E_{i+1} \rightarrow E_j).$$

(\star)

Clearly,

$$P(E_{i+1} \rightarrow E_j) = \frac{\mu_{i+1}}{\lambda_i + \mu_{i+1}}$$

and, by (\star),

$$\Pi_j^{(*)} = \Pi_j$$

for $j = 0, \ldots$. By substitution of these expressions into (\star), and reduction, we find

$$\lambda_i \Pi_i = \mu_{i+1} \Pi_{i+1} \quad (j = 0, \ldots).$$

(\ddagger)
For a birth-and-death process with $n$ sources, suppose the arrival rate in state $j$ ($E_j$), $j = 0, 1, \ldots, n-1$, depends only on the difference $n-j$, i.e., $\lambda_j[n] = \lambda(n-j)$, where $\lambda(n) > 0$ is any function. By Eq. (1), $\lambda_j[n] P_j[n] = \mu_j P_i[n]$ ($j = 0, 1, \ldots, n-2$), and as $\lambda_j[n] = \lambda(n-j) = \lambda_j[n-1]$, $\lambda_j[n-1] P_j[n] = \mu_j P_i[n]$ ($j = 0, 1, \ldots, n-2$). (2)

By Eq. (3.15) of Chapter 2, the outside observer's distribution in a system with $n-1$ sources will satisfy

$$\lambda_j[n-1] P_j[n-1] = \mu_{j+1} P_{j+1}[n-1] \quad (j = 0, 1, \ldots, n-2).$$

(3)

A comparison of (2) and (3) leads to the conclusion that

$$P_j[n] = R_j[n-1] \quad (j = 0, 1, \ldots, n-1)$$

(4)

for any finite-source birth-and-death process with $\lambda_j[n] = \lambda(n-j)$ and $\mu_j > 0$ for $j > 0$.

Chapter 5, Exercise 2

Burke's theorem.

*For the M/M/$\alpha$ queue in equilibrium, the sequence of service completion epochs follows a Poisson process (with the same parameter as the input process); that is, the output process is statistically the same as the input process.*

Let $T_j$ and $U_j$ be two arbitrary consecutive service completion epochs. Define $F_j(t)$ as the probability that simultaneously $T_j > t$ and the number of customers in system at $T_j + t$ equals $j$. Let $\mu_j = \mu j$ if $j \leq \alpha$, $\mu_j = \alpha \mu$ if $j > \alpha$.

\[ F_0(t+h) = F_0(t)[1-\lambda h] + o(h), \]

\[ F_j(t+h) = F_j(t)[1-(\lambda+\mu j)h] + F_{j-1}(t)\lambda h + o(h) \quad (j=1,\ldots). \]
(Chap. 5, Ex 2 a)

Hence,
\[ \frac{dF_j(t)}{dt} = -\lambda F_j(t), \]
\[ \frac{dE_j(t)}{dt} = -(\lambda + \mu(\eta)) F_j(t) + \lambda F_{j-1}(t). \]

with initial condition \( F_j(0) = \Pi_0^* \) for \( j = 1, 2, \ldots \).

It is easily found that \( F_j(t) = \Pi_j^* e^{-\lambda t}. \) We shall verify that the complete solution is
\[ F_j(t) = \Pi_j^* e^{-\lambda t} \quad (j = 0, 1, 2, \ldots). \]  (1)

(1) has been shown to produce the right answer for \( j = 0 \), and it clearly satisfies the initial condition for \( j = 1, 2, \ldots \). Thus it remains to demonstrate that the equation satisfies the differential-difference equations above for \( j = 1, 2, \ldots \).

Substitution of (1) into the appropriate differential-difference equation and some simple calculation and reduction yield
\[ \lambda \Pi_j^* = \mu(\eta) \Pi_{j-1}^* \quad (j = 1, 2, \ldots). \]  (2)

That this equation holds can be seen by making the substitution \( \Pi_j^* = \Pi_0^* \), which results in a special case of Eq. (1) of Ex. (i), or by making the substitution \( \Pi_j^* = F_j \), which results in the conservation-of-flow equation \( \lambda F_j = \mu(\eta) F_{j-1} \), \( j = 1, 2, \ldots \). We can therefore conclude that our Equation (1) indeed gives the desired probability \( F_j(t) \), for all \( t \) and \( j \).

Let \( F(t) \) denote the probability that \( T_2 > T_1 + t \), i.e.,
\[ F(t) = P(T_2 - T_1 > t). \]  By (1), and the definition of \( F_j(t) \),
\[ F(t) = \sum_{j=0}^{\infty} F_j(t) \]
\[ = \sum_{j=0}^{\infty} \Pi_j^* e^{-\lambda t} \]
\[ = e^{-\lambda t}. \]  (3)

Thus, the time separating two successive departures is exponentially distributed with the same mean as the interarrival times.
(Chap. 5, Ex. 2c)

Let \( x = T_2 - T_1 \) be the number of customers left behind by the departure of \( T_2 \). In equilibrium,

\[
P(\tilde{x}\geq x > t) = \int_t^\infty P_n(x) \mu^t g(x) \text{d}x
\]

\[
= \mu(x) \int_t^\infty \lambda e^{-\lambda x} \text{d}x
\]

\[
= \mu(x) \int_t^\infty \frac{e^{-\lambda x}}{\lambda} \text{d}x
\]

\[
= \int_t^\infty e^{-\lambda x} \text{d}x.
\]

Thus,

\[
P(\tilde{x}\geq x > t) = P(\tilde{x} = 1) P(x > t),\quad (4)
\]

which means that the length of the interdeparture interval \( x \) and the number of customers in the system at the start of the next interval are independent variables.

Denote by \( T_2 \) the departure epoch subsequent to \( T_2 \). By the Markov property

\[
P(T_2 - T_1 > t | \tilde{x} = 1, T_2 - T_1 > a) = P(T_2 - T_1 > t | \tilde{x} = 1).
\]

By (5) \( T_2 - T_1 \) may depend on \( T_2 - T_1 \) only through \( \tilde{x} \). But, by (4), \( T_2 - T_1 \) is independent of \( \tilde{x} \). Hence, \( T_2 - T_1 \) is independent of \( T_2 - T_1 \). An extension of this argument leads to the conclusion that all the interdeparture interval lengths are independent variables.

Remark. The particular form of the function \( \mu(x) \) used is of interest. It is worth noting that the whole line of proof applies to any birth-and-death process with birth rates \( \lambda_i = \lambda \) for all \( i \geq 0 \) and death rates \( \mu_i = 0 \) and \( \mu_i > 0 \) for \( i \geq 1 \), where the \( \lambda \) and \( \mu_i \)'s only work the condition for the existence of an equilibrium distribution. That is, the output process is a Poisson process with rate \( \lambda \) also in the general case, not just for an M/M/1 queue.

\(\square\)
Chapter 5, Exercise 3

Solve Exercise 13 of Chapter 2 by evaluating the integral.

\[ F_t(x) = \int_{\mathbb{R}_+} F_{t-x}(y) \, dy \, dF_A(x) \]

It is understood that \( F_{t-x}(y) = P(R_{t-x} \leq y) \), \( F_A(x) = P(A \leq x) \), and \( F_{t-x}(y) = P(T \leq t-x) \). The random variables \( R_t \) and \( A_x \) are independent; \( x \) and \( T \) are independent variables; and \( I_t = R_t + A_t \). Recall that

\[
\begin{align*}
F_{R_t}(y) &= 1 - e^{-\lambda y} \quad (y \geq 0), \quad \text{[by Eq. (530), Chap. 2]} \\
F_{A_x}(y) &= \begin{cases} 1 - e^{-\beta y} & (0 \leq y \leq t) \\ 0 & (y > t) \end{cases}, \quad \text{[by Eq. (533), Chap. 2]}
\end{align*}
\]

Hence,

\[
F_t(y) = \int_0^y F_{R_t}(x-y) \, dF_{A_x}(x) - \int_{y}^{\infty} [1 - e^{-\lambda x} - e^{-\beta x}] \, dF_A(x).
\]

\[ y < t \quad : \quad F_t(y) = \int_0^y \left[ 1 - e^{-\lambda x} - e^{-\beta x} \right] \, dx \]

\[ = 1 - e^{-\lambda y} - \lambda y e^{-\beta y} \]

\[ y \geq t \quad : \quad F_t(y) = \int_0^t \left[ 1 - e^{-\lambda x} - e^{-\beta x} \right] \, dx + \left[ 1 - e^{-\beta y} - e^{-\beta y} \right] \, e^{-\beta y} \]

\[ = 1 - e^{-\beta y} - \lambda e^{-\lambda y} \]

Thus, for all \( y \),

\[ F_t(y) = 1 - e^{-\beta y} - \lambda \min(y, t) e^{-\beta y}. \]

Chapter 5, Exercise 4

Let \( N \) be a nonnegative, integer-valued random variable

\[
\begin{align*}
\mathbb{E}[N] &= \int_0^\infty e^{-nt} \, dF(t) - \sum_{n=0}^\infty e^{-nt} \, P(N = n) \\
&= \sum_{n=0}^\infty P(N = n) (e^{-nt}) \\
&= g(e^{-t}).
\end{align*}
\]
Consider again the premise of Exercise 4 of Chapter 2, but let

\( X_i, i \geq 1, \) is a sequence of independent, identically distributed nonnegative random variables with distribution function \( F(t) = P(X \leq t) \) and Laplace-Stieltjes transform \( \Phi(t) = \int_0^\infty e^{-st} dF(t) \).

\( N \) is a nonnegative, integer-valued random variable with generating function \( g(s) = \sum_{n=0}^{\infty} P(N=n) s^n \). \( \{X_i\} \) and \( N \) are independent.

(a) Let \( S_0 = 0 \) if \( N = 0 \), and \( S_n = \sum_{i=1}^{N_n} X_i \) if \( n \geq 1 \). Clearly,

\[
P(S_n \leq t) = \sum_{n=0}^{\infty} P(N=n) P(S_n \leq t) = \sum_{n=0}^{\infty} P(N=n) F_n(t),
\]

where \( F_n(t) = 1 \) and \( F_n(t), n \geq 1, \) is the \( n \)-fold convolution of \( F(t) \).

Thus, letting \( \psi(s) \) denote the Laplace-Stieltjes transform of \( S_n \),

\[
\begin{align*}
\psi(s) &= \int_0^\infty e^{-st} dP(S_n \leq t) \\
&= \sum_{n=0}^{\infty} P(N=n) \int_0^\infty e^{-st} dF_n(t) \\
&= \sum_{n=0}^{\infty} P(N=n) [\Phi(s)]^n \quad \text{[by (5)]} \\
&= g(\Phi(s)).
\end{align*}
\]

(b) By differentiating \( \psi(s) \) twice we obtain

\[
\begin{align*}
\psi'(s) &= g'(\Phi(s)) \phi(s), \\
\psi''(s) &= g''(\Phi(s)) \phi(s) + g'(\Phi(s)) [\phi(s)]^2.
\end{align*}
\]

As \( \phi(0) = 1 \),

\[
\psi'(0) = g'(0) \phi(0), \quad \psi''(0) = g''(0) \phi(0) + g'(0) [\phi(0)]^2.
\]

Now, \( g'(0) = E(N), \ g''(0) = E(N^2) - E(N), \) and \( \phi(0) = -E(X), \phi''(0) = E(X^2). \) Hence,

\[
\psi''(0) = -E(N) E(X), \quad (1)
\]

and \( \psi''(0) = E(N) E(X^2) + [E(N^2) - E(N)] E(X), \) whereby

\[
\psi''(0) = E(N) \chi(X) + E(N^2) \chi(X), \quad (2)
\]
Mean and variance of $S_n$ may be derived from the Laplace-Stieltjes transform as follows:

\[
E(S_n) = E(S_0) - \psi(0),
\]

\[
V(S_n) = E(S_n^2) - [E(S_n)]^2 = \psi(0) - [\psi(0)]^2.
\]

By (1) and (2), then,

\[
E(S_n) = E(N)E(X),
\]

\[
V(S_n) = E(N)V(X) + V(N)E^2(X).
\]

We shall show in Section 5.8 that...

**W(t)** in the M/G/1 queue with service in order of arrival has Laplace-Stieltjes transform \( \omega(s) = \int_0^\infty e^{-st}dH(t) \) given by

\[
\omega(s) = \frac{s(1-e^s)}{\theta - \lambda[1-e^{-\lambda s}]},
\]

where \( \eta(s) = \int_0^\infty e^{-st}dH(t) \) is the Laplace-Stieltjes transform of the service-time distribution function \( H(t) \), with mean \( \tau = (0\rightarrow t) \), and \( \theta = \lambda \tau < 1 \), where \( \lambda \) is the arrival rate.

We shall determine the mean wait from the relation \( E(W) = \omega'(0) \). Differentiation of (1) results in

\[
\omega'(s) = \frac{\theta(1-e^s)}{[\theta - \lambda[1-e^{-\lambda s}]]^2}.
\]

where

\[
f(s) = \theta(1-\rho)[1-\eta(s)] + \eta'(s),
\]

\[
g(s) = (s - \lambda[1-\eta(s)])^2.
\]

Observe that \( f(0) = 0 \) and \( g(0) = 0 \). However, \( \omega'(0) = \lim_{s \to 0} \omega'(s) \) can be evaluated by a double application of l'Hopital's rule.
(Chap. 5, Ex. 6a)

First we calculate

\[ f'(s) = \lambda(1-\rho)s \eta(s), \]
\[ g'(s) = 2(1+\lambda \eta(s)(s-\lambda[s-\eta(s)])), \]

and, since \( f'(0) = 0 \) and \( g'(0) = 0 \), we differentiate again, obtaining

\[ f''(s) = \lambda(1-\rho) \eta'(s) + \lambda(1-\rho)s \eta''(s), \]
\[ g''(s) = 2(1+\lambda \eta(s)) \eta'(s) + \lambda \eta(s)(s-\lambda[s-\eta(s)]). \]

Hence,

\[ f''(0) = \lambda(1-\rho)(\eta + \sigma^2), \]
\[ g''(0) = 2(1-\rho)^2, \]

where \( \sigma^2 \) is the service-time variance, and we have used that \( \lambda \eta'(0) = \lambda(1-\rho) = -\rho. \) Finally, from \( E(W) = -\omega(0) = f''(0)/g''(0), \)

\[ E(W) = \frac{\sigma \rho}{2(1-\rho)(1 + \frac{\sigma^2}{\rho})}. \tag{2} \]

(b) When service-times are exponentially distributed with mean \( \mu^{-1} \), then the waiting-time distribution function \( W(t) \) is

\[ W(t) = \begin{cases} 0 & (t < 0), \\ 1 - e^{-\mu t} & (t \geq 0). \end{cases} \tag{3} \]

Thus,

\[ \omega(s) = \int_0^\infty e^{-st}dW(t) = e^{-st}P(W=0) + \int_0^\infty e^{-st}dW(t) = (1-\rho)(1-\rho) \int_0^\infty e^{-s(x+\lambda)}dxdt = (1-\rho) \frac{\sigma + \mu}{\sigma + \mu + \lambda}. \]

This result is in agreement with Equation (1), since in the case of an exponential service time distribution, we have \( \eta(s) = \mu/(\mu + s) \), so that (1) becomes

\[ \omega(s) = \frac{\lambda(1-\rho)}{\sigma + \lambda[s-\mu/(\mu + s)]} + (1-\rho) \frac{\sigma + \mu}{\sigma + \mu + \lambda}. \]
Chapter 5, Exercise 7

Show that if \( G(t) = 1 - e^{-\lambda t} \), then \( F(x) = 1 - e^{-\lambda x} \).

Since the interevent times have the exponential distribution with parameter \( \lambda \), \( G(t) = 1 - e^{-\lambda t} \), the mean of the interevent interval is

\[
\beta = \frac{1}{\lambda} \int_0^\infty x dG(x) = \lambda^{-1}.
\]

By (7.9) the equilibrium forward recurrence time has the distribution

\[
F(x) = \frac{1}{\beta} \int_0^x [1-G(t)] \, dt
= \frac{1}{\lambda} \int_0^x e^{-\lambda t} \, dt
= 1 - e^{-\lambda x},
\]

which is the same as the distribution of the interevent times.

Chapter 5, Exercise 8

Verify Equation (7.13).

The equilibrium forward recurrence distribution \( F(x) \) is given by (7.9) and has the Laplace-Stieltjes transform

\[
\varphi(s) = \frac{1}{\beta} \int_0^\infty e^{-sx} \, F(x) \, dx.
\]

Hence,

\[
\varphi(s) = \frac{1}{\beta} \int_0^\infty e^{-sx} \, [1 - G(x)] \, dx = \frac{1}{\beta} \int_0^\infty e^{-sx} \, dx = \frac{\frac{1}{\beta}}{s} = \frac{1}{s} \frac{\beta^2 + \alpha^2}{s}.
\]

Since both numerator and denominator equal zero for \( s = 0 \), \( \varphi(0) \) is evaluated by L'Hôpital's rule.

\[
\varphi(0) = \frac{1}{\beta} \lim_{s \to 0} \frac{s \varphi(s) + \varphi'(s) - \varphi(s)}{2s} = \frac{1}{\beta} \left( \frac{\beta^2 + \alpha^2}{2} \right) = \frac{\beta^2 + \alpha^2}{2}.
\]

As \( \beta^* = \int_0^\infty x dF(x) \) is determined by \( \beta^* = -\varphi(0) \), we have

\[
\beta^* = \frac{\beta}{\sqrt{2}} + \frac{\alpha^2}{2\beta}.
\]
Chapter 5, Exercise 4

(a) Show that

Define \( R_\delta = T_{ij} - t, I_\delta = T_{ij} - T_i \), where \( T_j \leq t < T_i \). Assume \( 0 \leq x \leq y \leq t \).

For given \( \{ (\gamma, \beta, \alpha) \} \) and \( t \) we have

\[
P(T_j \leq t < T_i, s \leq x, I_\delta \leq y) = \int_{t-y}^{t-x} [G(y) - G(t-x)] \, dP(T_j \leq t) + \int_{t-x}^{t} [G(t-x) - G(t-y)] \, dP(T_i \leq t)
\]

The formula is a simple consequence of the following observations:
(i) If \( T_j \leq t < T_i \), then the event \( \{ T_j \leq t, R_\delta \leq x, I_\delta \leq y \} \) cannot occur.
(ii) If \( t-y < T_j \leq t-x \), then the event will occur if and only if \( t-t_j \leq I_\delta \leq t-t_i \).
(iii) If \( t-x < T_j \leq t \), then the event will occur if and only if \( t-t_j \leq I_\delta \leq t-T_j \).

For \( y = 0 \) we have \( T_j \leq T_i \), so that, by (i), the probability of the event is zero.

Thus

\[
P(R_\delta \leq x, I_\delta \leq y) = \sum_{t=0}^{\infty} \left( \int_{t-y}^{t-x} [G(y) - G(t-x)] \, dP(T_j \leq t) \right) + \sum_{t=0}^{\infty} \left( \int_{t-x}^{t} [G(t-x) - G(t-y)] \, dP(T_i \leq t) \right)
\]

(b) Since \( m(3) = \sum_{t=0}^{\infty} P(T_j \leq t) \),

\[
P(R_\delta \leq x, I_\delta \leq y) = \int_{t-y}^{t-x} [G(y) - G(t-x)] \, dP(T_j \leq t) + \int_{t-x}^{t} [G(t-x) - G(t-y)] \, dP(T_i \leq t)
\]

Letting \( t \to \infty \) and using \( \lim_{t \to \infty} \frac{dP(T_j \leq t)}{dP(T_j \leq s)} = \frac{1}{P(T_j \leq s)} \), we obtain

\[
\lim_{t \to \infty} P(R_\delta \leq x, I_\delta \leq y) = \frac{1}{P(T_j \leq s)} \int_{t-y}^{t-x} [G(y) - G(t-x)] \, dP(T_j \leq s)
\]

\[
+ \frac{1}{P(T_i \leq s)} \int_{t-x}^{t} [G(t-x) - G(t-y)] \, dP(T_i \leq s)
\]
(Chap. 5, Ex. 9 b)

By the substitution \( t \to y \),
\[
\lim_{t \to \infty} P(R_t \leq x, I_t \leq y) = \frac{1}{\beta} \int_0^x \left[ (G_y-G(3))d3 + \frac{1}{\beta} \int_0^y \left[ (G_{3+y} - G(3))d3 \right] \right],
\]
which on further rewriting becomes
\[
\lim_{t \to \infty} P(R_t \leq x, I_t \leq y) = \frac{1}{\beta} \int_0^x G(y)dy - \frac{1}{\beta} \int_0^y G(3)dy \]
\[+ \frac{1}{\beta} \int_0^y (G(3)d3 - \frac{1}{\beta} \int_0^x G(3)dy).\]

Hence,
\[
\lim_{t \to \infty} P(R_t \leq x, I_t \leq y) = \frac{1}{\beta} \int_0^x [G(y) - G(3)]d3 \quad (0 < x < y). \quad (7.20)
\]

\( \Box \) By setting \( x = y \) in (7.20) we find
\[\lim_{t \to \infty} P(I_t \leq y) = \frac{1}{\beta} \int_0^y [G(y) - G(3)]dy \]
\[= \frac{1}{\beta} \int_0^y (G(y) - G(3))dy \]
\[= \frac{1}{\beta} \left. \int_0^y (G(y) - G(3)) \right|_0^y \]

whereby
\[
\lim_{t \to \infty} P(I_t \leq y) = \frac{1}{\beta} \int_0^y G(y)dy.
\]

By differentiation with respect to \( y \) and the subsequent substitution \( y = x \), we derive
\[
\lim_{t \to \infty} dP(I_t \leq x) = \frac{1}{\beta} x dG(x) \quad (7.19)
\]
which gives the equilibrium probability density of the covering interval.

It is also worth noting that by setting \( y = \infty \) in (7.20) we find again
\[
P(x) = \lim_{t \to \infty} P(R_t \leq x) = \frac{1}{\beta} \int_0^x [1 - G(3)]d3 \quad (7.9)
\]
(Chap. 5, Ex. 9, d)

1. We shall prove

\[ \lim_{t \to \infty} P(R_t \leq x) \mid I_t = y) = \frac{x}{y} \quad (0 \leq x \leq y). \]  

(7.21)

First (7.21) is proven under the assumption that \( G(3) \) has a discontinuity point at \( 3 = y \).

Obviously, in this case,

\[ \lim_{t \to \infty} P(R_t \leq x) \mid I_t = y) = \frac{\lim_{t \to \infty} P(R_t \leq x, I_t = y)}{\lim_{t \to \infty} P(I_t = y)}. \]  

(*)

By (7.20),

\[ \lim_{t \to \infty} P(R_t \leq x, I_t = y = \frac{1}{\rho} \int_0^1 dG(y) \int_0^1 dy = \frac{x \cdot dG(y)}{\rho}, \]

and setting \( x = y \) in the above equation we derive

\[ \lim_{t \to \infty} P(I_t = y) = \frac{y \cdot dG(y)}{\rho}. \]

Substitution of the last two expressions into (*) proves (7.21) in the discontinuous case.

Next we prove (7.21) under the assumption that \( G(3) \) is continuous and differentiable at \( 3 = y \). Then

\[ \lim_{t \to \infty} P(R_t \leq x) \mid I_t = y) = \lim_{t \to \infty} P(R_t \leq x, y \leq I_t \leq y + dy) = \frac{\lim_{t \to \infty} P(R_t \leq x, y \leq I_t \leq y + dy)}{\lim_{t \to \infty} P(y \leq I_t \leq y + dy)}. \]  

(2 *)

By (7.20),

\[ \lim_{t \to \infty} P(R_t \leq x, y \leq I_t \leq y + dy) \cdot \frac{x \cdot dG(y)}{\rho} \cdot dy = \frac{x \cdot dG(y)}{\rho} \cdot dy, \]

and by setting \( x = y \) in this equation we find

\[ \lim_{t \to \infty} P(y \leq I_t \leq y + dy) = \frac{y \cdot dG(y)}{\rho} \cdot dy. \]

Substitution into (**) once more results in (7.21).
(Chap. 5, Ex. 9 e)

6. The equivalence of (7.20) and (7.22) is established by the following sequence of pairwise equivalent equations leading from (7.20) to (7.22):

\[
\begin{align*}
\lim_{t \to \infty} P(R_t \leq x, T_x \leq y) &= \frac{1}{\lambda} \int_0^y [G(y) - G(y)] dy \quad (0 \leq x \leq y) \quad (7.20) \\
\lim_{t \to \infty} dP(R_t \leq x, T_x \leq y) &= \frac{1}{\lambda} dG(y) dx \quad (0 \leq x \leq y) \\
\lim_{t \to \infty} dP(R_t \leq x, A_t \leq y) &= \frac{1}{\lambda} dG(x+y) dx \\
\lim_{t \to \infty} P(R_t \geq x, A_t \geq y) &= \int_0^\infty \int_0^y \frac{1}{\lambda} dG(x+y) dx dy \\
\lim_{t \to \infty} P(R_t \geq x, A_t \geq y) &= \frac{1}{\lambda} \int_0^y \int_0^\infty [1 - G(x+y)] dx dy \\
\lim_{t \to \infty} P(R_t \geq x, A_t \geq y) &= \frac{1}{\lambda} \int_0^y \int_0^\infty [1 - G(x)] dx dy \quad (7.22)
\end{align*}
\]

7. Observe that, by (7.22), the probability that either \( R_t = 0 \) or \( A_t = 0 \) is zero, since \( \lim_{t \to \infty} P(R_t \geq 0, A_t \geq 0) = \frac{1}{\lambda} \int_0^\infty [1 - G(x)] dx = 1 - e^{-\lambda x} \). Substituting \( G(x) = 1 - e^{-\lambda x} \) in (7.22) it is found that

\[
\lim_{t \to \infty} P(R_t \geq x, A_t \geq y) = e^{-\lambda x \varphi(x)} \quad (\ast)
\]

Now,

\[
\lim_{t \to \infty} P(R_t \geq x) = \lim_{t \to \infty} P(R_t \geq x, A_t \geq 0) = \lim_{t \to \infty} P(R_t \geq x, A_t \geq 0)
\]

By (\ast),

\[
\lim_{t \to \infty} P(R_t \geq x) = e^{-\lambda x}.
\]

Similarly,

\[
\lim_{t \to \infty} P(A_t \geq y) = e^{-\lambda y}.
\]

As \( \lim_{t \to \infty} P(R_t \geq x, A_t \geq y) = \lim_{t \to \infty} P(R_t \geq x) \lim_{t \to \infty} P(A_t \geq y) \), the conclusion is that \( R_t \) and \( A_t \) are, in the limit, independent exponential variables.
Chapter 5, Exercise 10

"Customers arrive at a single server..."

\( G(t) \) = interarrival time distribution function
\( H(t) \) = service-time distribution function
\( F(x) \) = cycle-time distribution function

\[ F(x) = \int_0^x P(R \leq x-t) \, dH(t), \]

where \( t = 0 \) is the time service starts. By Eq. (7.14),

\[ F(x) = \sum_{j=1}^{\infty} \int_0^x \left[ 1 - G(x-y) \right] \, dG^*(y) \, dH(t). \]

Interchanging the order of integration and summation we find

\[ F(x) = \sum_{j=1}^{\infty} \int_0^x \left[ 1 - G(x-y) \right] \, dG^*(y) \, dH(t). \]  \( (1) \)

Henceforth we assume \( H(t) = 1 - e^{mt} \) \( (t > 0) \).

\( B \)
Substitution of \( dH(t) = \mu e^{-mt} dt \) and change of the order of integration give

\[ F(x) = \sum_{j=1}^{\infty} \int_0^x \left[ 1 - G(x-y) \right] \, \mu e^{-mt} \, dG^*(y). \]

Hence,

\[ F(x) = \sum_{j=1}^{\infty} \int_0^x \left[ 1 - G(x-y) \right] \, e^{-mx} \, dG^*(y). \]

By differentiation w.r.t. \( x \) we find

\[ dF(x) = \sum_{j=1}^{\infty} \int_0^x \left[ - G(x-y) \right] \, e^{-mx} \, dG^*(y) = -\sum_{j=1}^{\infty} \int_0^x \left[ 1 - e^{-mx} \right] \, dG^*(y). \]

With \( G(0) = 0 \), this expression can be written

\[ dF(x) = \sum_{j=1}^{\infty} dG^*(y) \left[ e^{-mx} \, dG^*(y) - e^{-mx} \, dG^*(y) + \frac{1}{m} \, dG^*(y) \right], \]

where

\[ dG^*(y) = \int_y^\infty dG(x-y) \cdot e^{-mx} \, dG^*(y). \]
(Chap. 5, Ex. 16 (b))

Hence,

\[
\phi(s) = e^{-\mu s} \int_0^\infty e^{-\lambda x} f(x) \, dx
\]

\[
= \frac{\phi(s)}{s} \left( e^{-\mu s} \int_0^\infty e^{-\lambda x} G(x) \, dx - \frac{\phi(s)}{s} \int_0^\infty e^{-\lambda x} G''(x) \, dx \right)
\]

Introducing \( \gamma(s) = \int_0^\infty e^{-\lambda x} G(x) \, dx \) we derive

\[
\phi(s) = \psi(s) - \frac{\psi(s) + \gamma(s + \mu)}{1 - \gamma(s + \mu)},
\]

which reduces to

\[
\phi(s) = \psi(s) - \frac{\psi(s) + \gamma(s + \mu)}{1 - \gamma(s + \mu)}. \tag{2}
\]

\[\begin{align*}
\phi''(s) &= \frac{\psi'(s)}{1 - \gamma(s + \mu)} - (1 - \gamma(s)) \frac{\psi'(s + \mu)}{(1 - \gamma(s + \mu))}.
\end{align*}\]

The mean cycle time \( \alpha \) is given by \( \alpha = -\phi'(0) \). Thus,

\[
\alpha = \frac{-\phi'(0)}{1 - \gamma(M)} . \tag{3}
\]

\[\begin{align*}
e & \quad \text{Evidently the equilibrium probability} \quad P \text{ that the server is busy equals the rate of mean service time to mean cycle time.} \quad \text{That is} \quad P = \mu^{-1} / \alpha. \quad \text{By (3),}
\end{align*}\]

\[
P = \frac{-\psi'(0)}{1 - \gamma(M)} \frac{1}{1 - \gamma(M)}. \tag{4}
\]

\[\begin{align*}
e & \quad \text{By Eq. (14) of Chapter 2, the blocking probability is} \quad P = E(N)/(1 + E(N)), \quad \text{where} \quad N \text{ is the number of arrivals during a random service initiated by an arrival at an idle server.} \quad N \text{ is the number of failures in a sequence of Bernoulli Trials.}
\end{align*}\]

A failure in the occurrence of another arrival before service completion and has probability \( q = 1 - p = e^{-\lambda x} G(x) = \gamma(x) \). Thus, \( N \) has the geometric distribution and \( E(N) = q/p = \gamma(x)/(1 - \gamma(x)). \quad \text{Hence,} \quad P = \gamma(M). \tag{5}
\]
Equations (4) and (5) imply
\[ P = [I - \gamma'(0)M]^{-1} [I - \Pi]. \tag{6} \]

It is easy to see that \( P \) equals the carried load. Now, \( -\gamma'(0) \) is the mean interarrival time. Hence, letting \( \lambda \) mean arrival rate, \( \lambda = [\gamma'(0)]^{-1} \). Thus, \( P = \frac{\lambda}{\lambda - \mu} \). That is, carried load \( P \) equals offered load \( \lambda \mu \) times acceptance probability \( \frac{1}{1 - \lambda/\mu} \).

Suppose \( G(t) = 1 - e^{-\mu t} \). Then \( \gamma(s) = \lambda / (\lambda + s) \).

By (5) and (6),
\[ \Pi = \frac{\lambda}{\lambda + \mu}, \]
\[ P = \frac{\lambda}{\lambda} \left( 1 - \Pi \right) = \frac{\lambda}{\lambda} \left( 1 - \frac{\lambda}{\lambda + \mu} \right) = \frac{\lambda}{\lambda + \mu}. \]

In this case, then, (ii) \( P = \Pi \), as anticipated.

By (3),
\[ \phi(s) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + s} = \frac{\lambda}{\lambda + \mu} \cdot \frac{\lambda}{\lambda + \mu}\ .\]

Since \( \phi(s) \) is the product of two Laplace transforms of exponential distributions we can conclude that statement (ii) is true.

Suppose interarrival times are of constant length \( \tau \). Then
\[ F(x) = 1 - e^{-\tau x} \quad \text{(for } x > 0, \text{ for } j = 0, \lambda/\mu), \tag{7} \]
and
\[ \phi(s) = \int_0^\infty e^{-sx} dF(x) = \int_0^\infty e^{-sx} [e^{-\tau x} - e^{-\tau x} + e^{-\tau x}] \]
\[ = (e^{\tau \lambda}) \int_0^\infty e^{-sx} [e^{-\tau x} - e^{-\tau x} + e^{-\tau x}] \frac{e^{-sx} - e^{-sx}}{1 - e^{-sx}} \tau. \]

We would get the same result from an application of Eq. (2).

In the present case, \( \gamma(s) = \frac{\lambda}{\lambda + s}, G(t) = e^{-\mu t} \). Substitution into (2) yields the formula above.
Chapter 5, Exercise 11

"Customers arrive according to a renewal process..."

a) Suppose the arrival stream at a server is a renewal process, and that service times are exponential, and blocked customers are cleared. Let \( \{T_i\} \) \( (i=1,2,\ldots) \) denote the resultant sequence of overflow epochs.

\( T_i \) is completely determined by (i) remaining service time at \( T_{i-1} \), (ii) the sequence of future service times, (iii) the sequence of future interarrival intervals. All these variables are independent of the process up until \( T_i \) and also independent of \( j \).

It follows that the sequence of interevent intervals \( \{T_{i+1}-T_i\} \) are independent, identically distributed random variables. That is, the overflow stream is a renewal process.

We conclude that, under the assumptions of this exercise, the overflow stream from the \( i \)-th ordered server \((i=1,2,\ldots)\) is a renewal process.

b) Let \( G(t) \) be the distribution function of times between successive overflows from the \( i \)-th server. Choose an arbitrary overflow epoch of the \( i \)-th server. Let \( X \) be the time until next arrival (i.e. overflow from the \((i+1)\)-th server), and let \( Y \) be the time until next overflow from the \( i \)-th server. Then, for \( x \geq 0 \),

\[
P(Y \leq t | X=x) = e^{-mx} + (1-e^{-mx})G(t-x),
\]

with \( G(0) = 0 \), which implies \( G(0) = 0 \). This is so, since for \( X=x \), the event \( Y \leq t \) will occur if (a) service is completed after next arrival \((x \text{ time units later})\), having probability \( e^{-mx} \), or if (b) service is completed before next arrival and an overflow from the \( i \)-th server takes place during the \( x \)-time interval \((x,t)\), having probability \((1-e^{-mx})G(t-x)\). Observe that the time from start of service until next overflow from the \( i \)-th server has the same distribution as the inter-overflow times of the server.

Clearly,

\[
P(Y \leq t) = \int_0^t P(Y \leq t | X=x) \, dP(X=x).
\]
Now, \( P(Y \leq t) = G_i(t) \) and \( P(X \leq x) = G_{ic}(x) \). Hence,

\[
G_i(t) = \int_0^t \left[ e^{-\lambda x} + (1 - e^{-\lambda}) G_i(t-x) \right] dG_{ic}(x) \quad (i = 1, 2, \ldots)
\]

as asserted.

Differentiation of Eq. (1) leads to

\[
dG_i(t) = e^{\alpha t} dG_{ic}(t) + \int_0^t (1 - e^{-\lambda x}) dG_i(t-x) dG_{ic}(x).
\]

It follows that

\[
y_i(s) = \int_0^\infty e^{-st} dG_i(t)
\]

\[
= \int_0^\infty e^{-(s+u)t} dG_{ic}(t)
\]

\[
+ \int_0^\infty e^{-st} \int_0^t dG_i(t-x) dG_{ic}(x)
\]

\[
- \int_0^\infty e^{-st} \int_0^t dG_i(t-x) e^{-\lambda x} dG_{ic}(x)
\]

\[
= \int_0^\infty e^{-(s+u)t} dG_{ic}(t)
\]

\[
+ \int_0^\infty e^{-st} dG_i(t) \cdot \int_0^\infty e^{-\lambda x} dG_{ic}(x)
\]

\[
- \int_0^\infty e^{-st} dG_i(t) \cdot \int_0^\infty e^{-\lambda x} dG_{ic}(x)
\]

Hence,

\[
y_i(s) = y_i(s+\alpha \mu) + y_i(s) y_i(s) - y_i(s) y_i(\alpha \mu),
\]

from which is obtained the recurrence equation

\[
y_i(s) = \frac{y_i(s+\alpha \mu)}{1 - y_i(s) y_i(\alpha \mu)} \quad (i = 1, 2, \ldots)
\]
Chapter 5, Exercise 12

The M/G/1 queue with server vacation times.

Let $P(j)$ be the probability that $j$ customers arrive during a single, arbitrary vacation, and define

$$f(t) = \sum_{j=0}^{\infty} P(j) e^{jt}.$$

Let $X$ denote the number of customers $(X \geq 1)$ who arrive during the vacation period. Clearly, $P(X=0) = 0$ and, for $j \geq 1,$

$$P(X=j) = P(j)/[1-P(0)].$$

Letting $P(a) = \sum_{j=0}^{\infty} P(X=j) a^j$ be the probability generating function for $X$, we derive

$$\hat{P}(a) = \frac{P(0) e^a}{1 - P(0)}.$$  \hfill (8)

Proceeding as in the analysis of the M/G/1 queue without vacation, we find the following system of equations:

$$\hat{\pi}_j = \gamma_j \sum_{k=0}^{\infty} \hat{\pi}_k + \sum_{k=0}^{j-1} \gamma_k \hat{\pi}_{k+1}$$  \hfill (8.20)

Here, $\gamma_j = P(X+Y=j+1)$, where $X$ is the number of customers arriving during the vacation period and $Y$ is the number of customers arriving during the initial service after vacation.

Consider for a moment the p.g.f. of $X+Y$. Since $X$ and $Y$ are independent variables

$$\hat{P}(a) = \sum_{j=0}^{\infty} P(j) a^j = \frac{P(0) e^a}{1 - P(0)},$$  \hfill (8.21)

with $h(t) = \sum_{j=0}^{\infty} P(j) a^j$ being the p.g.f. of the number of arrivals during an arbitrary service time. Note, $\gamma_0 = 0$, as $X \geq 1$.

Substitution of Eq. (8.20) into the generating function

$$\hat{g}(a) = \sum_{j=0}^{\infty} \hat{\pi}_j a^j$$

results in

$$\hat{g}(a) = \sum_{j=0}^{\infty} \gamma_j \hat{\pi}_j a^j + \sum_{j=0}^{\infty} \sum_{k=0}^{j-1} \gamma_k \hat{\pi}_{k+1} a^j.$$  \hfill (8.23)
Now, by (**), and the fact that $\tau_0 = 0$,
\[
\sum_{k=0}^{\infty} \tau_k x^k = z^{-r} \sum_{k=0}^{\infty} \tau_k z^k = z^{-r} \hat{g}(z) h(z),
\]
and, furthermore,
\[
\sum_{k=0}^{\infty} \sum_{i=1}^{1} \tau_k x^i = x^{-r} \sum_{k=0}^{\infty} \tau_k x^i = x^{-r}(\hat{g}(z) - \hat{\tau}_0) h(z). \tag{8.29'}
\]
Hence,
\[
\hat{g}(z) = \hat{\tau}_0^* x^{-r} \hat{\tau}_0 h(z) + x^{-r}(\hat{g}(z) - \hat{\tau}_0) h(z). \tag{8.29''}
\]
By (*), \( \hat{f}(z) = 1 + (f(z) - 1)/(1-P(0)) \). Substituting this into (8.29'') and simplifying, we obtain
\[
\hat{g}(z) = \hat{\tau}_0^* x^{-r} \frac{\hat{f}(z) - 1}{1-P(0)} h(z) + x^{-r} \hat{g}(z) h(z).
\]
Solving for \( \hat{g}(z) \), we find
\[
\hat{g}(z) = \frac{\hat{f}(z) - 1}{1-P(0)} \frac{\hat{\tau}_0^*}{1-P(0)} \hat{h}(z). \tag{8.30'}
\]
A utilization of the condition \( \hat{g}(1) = 1 \) gives
\[
\hat{\tau}_0^* = \frac{(1-P(0))}{\hat{\tau}_0}. \tag{8.32'}
\]
with \( \rho = \lambda \tau = h(0) \). When this expression and \( h(z) = \eta(1-\lambda z) \) are substituted into (8.30'), we get
\[
\hat{g}(z) = \frac{[(f(z) - 1) \eta(1-\lambda z)]}{1-P(0) \eta(1-z)} \frac{1-P(0)}{f(z)} \tag{1}
\]
A comparison of (1) and (8.12) shows that
\[
\hat{g}(z) = \frac{1-P(0)}{f(z)\eta(1-z)} \hat{g}(z) \tag{2}
\]
where \( g(z) \) is the probability generating function of the number of customers left behind by an arbitrary departing customer in the corresponding equilibrium M/G/1 system, in which the server never goes on vacation.
(Chap 5, Ex 12 b)

b) We shall prove that $g(z) = g(z) \iff f(z) = P(0) + P(1) z$.
This means that $g(z) = g(z)$ implies that no more than one customer will ever arrive during a vacation. As the arrival process is Poisson, $g(z) = g(z)$ therefore automatically rules out the possibility that vacation length is independent of the arrival process. Our explanation of the condition $g(z) = g(z)$ is that a vacation, if not already over, will be interrupted the moment an arrival takes place.
By (2), $g(z) = g(z) \iff f(z) = f(0) (z-1)$. Hence, it will suffice to show that $f(z) = f(z) (z-1) \iff f(z) = P(0) + P(1) z$.

Necessity ($\Rightarrow$). Assume $f(z) = f(z) (z-1)$ for all $z$. Then $f(z) = (1-P(0)) + P(1) z$. Since $f(z) = \sum_{n=0}^{\infty} P(n) z^n$, it follows that $P(0) = 1-P(1)$ and $P(1) = P(1)$. Thus, $f(z) = P(0) + P(1) z$.

Sufficiency ($\Leftarrow$). Assume $f(z) = P(0) + P(1) z$. Then $P(0) = 1-P(1)$ so that $f(z) = (1-P(0)) + P(1) z$. Hence, $f(z) = f(z)$ and, in particular, $P(0) = P(1)$. Thus, $f(z) = f(z) (z-1)$ for all $z$.

c) By Eq. (3.3), $\hat{g}(z) = \hat{f}(z)$ for all $z \geq 0$. Consequently,
$$\hat{g}(z) = \hat{f}(z) = \sum_{n=0}^{\infty} \frac{(P(0) - 1)}{P(0)} z^n,$$
where $g(z)$ is given by (1). By (3.3), $\hat{g}(z) = \hat{f}(z)$,
$$\hat{f}(z) = (1-P(0)) \frac{P(0)}{P(1)}$$
($\hat{f}(z)/P(0)$ is mean number of customers by end of vacation.)

d) Differentiation of Eq. (2) gives
$$g'(z) = \frac{P(0) - 1}{P(0)} g(z) + \frac{g(z) - g(z)z - P(0)}{P(0)} (z-1),$$
whereby
$$g'(z) = \lim_{z \to 1} \frac{g(z) - g(z)z - P(0)}{P(0)} (z-1),$$
(Chap. 5, Ex. 12 d)

A single application of l'Hospital's rule yields

\[
\tilde{g}'(s) = \lim_{s \to 0^+} \frac{\tilde{f}(s)}{s} = \frac{1}{s} \lim_{s \to 0^+} \left( s \tilde{f}(s) - \tilde{f}(s) - \tilde{f}'(s) \right)
\]

Hence,

\[
\tilde{g}'(s) = \tilde{f}(s) - \frac{\tilde{f}(s)}{s} \tilde{f}'(s).
\]  (5)

\[\text{Letting } \tilde{f}(s) \text{ be the Laplace-Stieltjes transform of the sojourn time, we have:}
\]

\[
\tilde{f}(s) = \tilde{\omega}(s) \eta(s),
\]  (8.78)

\[
\tilde{g}(s) = \tilde{\phi}(\lambda - \lambda s).
\]  (8.79)

Thus,

\[
\tilde{g}(s) = \tilde{\omega}(s) \eta(s) (\lambda - \lambda s).
\]

Insertion of this expression into (7) leads to

\[
\tilde{\omega}(\lambda - \lambda s) = \frac{\tilde{f}(s) - \tilde{f}'(s)}{s \eta(s) - \lambda s \eta'(s)}.
\]  (8.80)

Setting \( s = \lambda - \lambda s \), we derive the analogue of (8.38)

\[
\tilde{\omega}(s) = \frac{1 - \tilde{f}(1 - \tilde{f})}{s - \lambda(1 - \tilde{f})} \frac{\lambda}{\tilde{f}(1)}.
\]  (4)

\[\text{In the case that vacation lengths are independent of the arrival process, evidently the probability of waiting equals } 1.
\]

\[\text{The L.-S. transform of the distribution function of the waiting time } \tilde{W} \text{ is } \tilde{\omega}(s), \text{ given by (4). By comparison of (4) and (8.38) we find the relation}
\]

\[
\tilde{\omega}(s) = \frac{1 - \tilde{f}(1 - \tilde{f})}{s} \frac{\lambda}{\tilde{f}(1)} \tilde{\omega}(s).
\]

Hence,

\[
\tilde{\omega}(s) = \frac{1 - \tilde{f}(1 - \tilde{f})}{s} \frac{\lambda}{\tilde{f}(1)} \tilde{\omega}(s) + \frac{\tilde{f}(1 - \tilde{f})}{s} \frac{\lambda}{\tilde{f}(1)} \tilde{\omega}(s),
\]
\[ \hat{\omega}(0) = \lim_{n \to \infty} \frac{n(n-1)}{2} \frac{\omega(n)}{n} + \lim_{n \to \infty} \frac{\sum_{i=1}^{n} i^2 (n-i)}{n^3} \frac{\omega(n)}{n} \]

A single application of I'Hospital's rule produces
\[ \hat{\omega}(0) = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \frac{\omega(n)}{n} + \lim_{n \to \infty} \frac{\sum_{i=1}^{n} i^2 (n-i)}{n^3} \frac{\omega(n)}{n} \]

by which
\[ \hat{\omega}(0) = \omega(0) - \frac{\omega'(0)}{2} \frac{1}{\sqrt{n}} \]

New, \( E(\hat{N}) = -\omega(0) \) and \( E(N) = -\omega(0) \). It follows that
\[ E(\hat{N}) = E(N) + \frac{\omega'(0)}{2} \frac{1}{\sqrt{n}} \]

where \( E(N) \) is given by (8.39).

b) Suppose \( P(a) = a^2 \) for all \( j \geq 1 \). That is, with probability \( a \) exactly \( j \) customers arrive during a vacation, and arrival no. \( j \) signals the end of the vacation. Obviously, the length of a vacation depends on future arrivals.

Eq. (1), (2), and (3), hold for \( P(a) = a^2 \) for all \( j \geq 1 \), since the equations were derived without a requirement of independence between vacation length and arrival process.

Eq. (4) results from (1), (8.42), and (8.45). Of these, (1) and (8.42) hold whether or not vacation length depends on the arrival process. However, (8.45) is valid only if service time is independent of the arrival process. For \( P(a) = a^2 \) with \( j > 1 \) this condition will be met for every customer, but if \( j \geq 2 \), then in the case of arrivals \( a_1, a_2, \ldots, a_j \), the time until the end of vacation will depend on the future arrival epochs.

We conclude that Eq. (4) is valid for \( j = 1 \), but not for \( j \geq 2 \), when \( P(a) = a^2 \). In the case \( P(a) = a^2 \), Eq. (4) reduces to the Bollaczek-Khintchine formula (8.38), as it should, when \( P(1) = 1 - \frac{1}{n} \) and \( P'(1) = 1 \) are inserted.

Eq. (5) follows from (3) after application of \( L = \lambda \mathcal{W} \).

(5) is therefore valid for all \( j \geq 1 \) when \( P(a) = a^2 \).
Chapter 5, Exercise 13

'Derivation of the P-K formula by the method of collective marks.'

Note that in this exercise, the same notation is used for a time interval and its length. For instance, $W_k$ will denote both the time interval during which the kth customer waits for service and the length of that time interval, the waiting time. No confusion should arise as the meaning is clear from the context.

a) We consider an M/G/1 queue with order-of-arrival service. The kth arriving customer is here the same as the kth departing customer, so we may also speak of the kth customer.

Let $P_{k}^{(m)}$ be the probability that the kth customer will leave j customers behind, namely those customers who arrive during his sojourn time, and define the generating function

$$g_k(z) = \sum_{j=0}^{\infty} P_{k}^{(m)} z^j$$

Now, imagine that each arriving customer is marked with probability 1/2 and left unmarked with probability 1/2. Clearly, by the theorem of total probability, $g_k(z)$ may be interpreted as the probability that no marked customers arrive during the sojourn time of the kth customer.

b) \( T_k \) = sojourn time of the kth customer (or corresponding time interval)

\( W_k \) = waiting time of the kth customer (or corresponding time interval)

\( C_k \) = { the kth customer is marked }

\( C_k^c \) = { the kth customer is not marked }

\( M(X) \) = { no marked customers arrive during time interval X }

It follows from the above definitions and our assumption of order-of-arrival service that

\[ \{ M(T_k), C_{k_a} \} \Leftrightarrow \{ W_{k_1} - 0, C_{k_2} \} \]

\[ \{ M(T_k), C_{k_a}^c \} \Leftrightarrow \{ M(W_{k_1}), C_{k_2} \} \]
(Chap. 5, Ex. 13 b)

Also, since a customer's probability of being marked is 1-z whatever his waiting time and markings of other customers,

\[
P(W_{s+1} = 0, C_{s+1}) = P(W_{s+2} = 0) | P(C_{s+1})|
\]
\[
P(M'(W_{s+1}, C_{s+1}) = P(M'(W_{s+2}))| P(C_{s+1})|.\]

Since \( P(M'(W_{s+1})) = P(M'(W_{s+1}, C_{s+1}) + P(M'(W_{s+1}, C_{s+1}^c)) \), we have

\[
P(M'(W_{s+1})) = P(W_{s+2} = 1) (1-z) + P(M'(W_{s+2})) z.
\]

\(c\) Denote by \( \phi_s(a) \) and \( \omega_s(a) \) the Laplace-Stieltjes transforms of the distribution functions of \( T_a \) and \( W_a \), respectively. By the definition of \( \phi_s(a) \), the interpretation in part (a), and Equation (1.10),

\[
P(M'(T_a)) = \phi_s(a) = \phi_s(\lambda - \lambda_s).
\]

Similarly,

\[
P(M'(W_a)) = ... = \omega_s(\lambda - \lambda_s).
\]

\(d\) By parts (b) and (c),

\[
\phi_s(\lambda - \lambda_s) = P(W_{s+2} = 0) (1-z) + \omega_s(\lambda - \lambda_s) z.
\]

The substitution \( \lambda - \lambda_s = s \) gives

\[
\phi_s(a) = P(W_{s+2} = 0) \frac{s}{\lambda} + \omega_s(a) (1 - \frac{s}{\lambda}).
\]

Introducing \( \phi_s(a) = \omega_s(a) \eta(a) \), letting \( k \to 0 \), assuming that \( \lim \omega_s(\lambda - \lambda_s) = \omega_s(\lambda - \lambda_s) \) and \( \lim P(W_{s+2} = 0) \), and solving for \( \omega_s(a) \),

we find

\[
\omega_s(a) = \frac{s}{s - \lambda [1 - \eta(\lambda_s)]} P(W = 0).
\]

\(e\) Finally, utilizing \( \omega_s(a) = 1 \) we derive \( P(W = 0) = 1 + \lambda [1 - \eta(\lambda_s)] - 1 - \eta = 1 - \rho \). Once more we obtain the Pabstek-Knudelhine formula

\[
\omega_s(a) = \frac{s [1 - \rho]}{s - \lambda [1 - \eta(\lambda_s)]} (838).
\]
Chapter 5, Exercise 14

The M/G/1 queue from the viewpoint of arrivals.

Definitions

\( N = \) number of customers in the system just prior to an arbitrary arrival epoch \( T_0 \)

\( R = \) remaining service time at \( T_0 \)

\[ \pi_j = P(N = j) \quad (j = 0, 1, \ldots) \]  
(1)

\[ \pi_j(x) = P(R \leq x, N = j) \quad (x \geq 0, j = 1, 2, \ldots) \]  
(2)

\[ \psi_j(s) = \int_0^\infty e^{-sx} \pi_j(x) \, dx \]  
(3)

\[ \nu(j) = \frac{\psi_j}{\pi_j} \psi_j(s) s^j \]  
(4)

Main results

\[ u(\alpha, \beta) = \frac{\pi_2 \alpha (1-\beta)}{\beta - \eta(1-\beta) - \lambda - \lambda (1-\beta)} \]  
[see part e.2.] \hspace{1cm} (5)

\[ \pi_0 = 1 - \rho \quad (\rho = \lambda \mu) \]  
[see parts.] \hspace{1cm} (6)

where \( \eta(w) \) is the Laplace-Stieltjes transform of the service-time distribution function \( H(x) \), \( \gamma \) is the mean service time, and \( \lambda \) is the customer arrival rate. Inversion of (5) gives

\[ \sum_{j=1}^{\infty} \frac{1}{j!} \pi_j(x) x^j = \frac{(1-\rho) \lambda}{\eta(\lambda-\beta) - \lambda \beta} \int_0^\infty e^{-t(x-\beta)} \left[ H(t(x-\beta)) - H(t) \right] dt \]  
[see parts.] \hspace{1cm} (7)

\[ u(\alpha, 1) = \pi_0 \lambda \lim_{s \to 0} \frac{1 - \alpha}{1 - \eta(1-\beta) - \lambda + \lambda \eta(1-\beta)} \lim_{s \to 0} \frac{\psi(s, 1) - \psi(s, 1-\beta)}{s} \]

\[ = \pi_0 \lambda \frac{-\psi(0)}{1 + \lambda \eta(0)} = \pi_0 \frac{\lambda}{1 + \lambda \eta(0)} - \pi_0 \frac{\lambda}{1 - \rho} \]

\[ = \pi_0 + \sum_{j=1}^{\infty} \frac{d j}{j!} - \pi_0 + \sum_{j=1}^{\infty} \frac{d j}{j!} \psi(0) \]

\[ = \pi_0 + u(\alpha, 1) = \pi_0 (1 + \frac{\rho}{1 - \rho}) \]

Thus, Eq. (6), \( \pi_0 = 1 - \rho \), follows from Eq. (5) and \( \sum \pi_j = 1 \).
(Chap. 5, Ex. 14 b)

By Eq. (77),

\[ \sum_{i=1}^{\infty} \Pi_i(x) = \lim_{x \to \infty} \sum_{i=1}^{\infty} \Pi_i(x) e^x = (1-\rho) \lim_{x \to \infty} \frac{1}{\eta(\lambda-\lambda x)} \int_0^\infty [H(3+\lambda x) - H(3)] dx \]

\[ = (1-\rho) \lambda \frac{1}{1 + \lambda \eta \rho} \left( \int_0^\infty [1-H(3)] dx - \int_0^\infty [1-H(3)] dx \right). \]

Hence,

\[ \sum_{i=1}^{\infty} \Pi_i(x) = \lambda \int_0^\infty [1-H(3)] dx, \]

so that

\[ P(R \times N \geq 1) = \frac{P(R \times N \geq 1)}{P(N \geq 1)} = \frac{\sum_{i=1}^{\infty} \Pi_i(x) \lambda}{\sum_{i=1}^{\infty} \Pi_i(x) \lambda} = \frac{\lambda \int_0^\infty [1-H(3)] dx}{\lambda \int_0^\infty [1-H(3)] dx} = 1. \]

Since the mean service time may be expressed as \( \tau = \int_0^\infty [-H(3)] dx \),

\[ P(R \times N \geq 1) = \frac{1}{\tau} \int_0^\infty [1-H(3)] dx. \]

This might have been anticipated by considering (28); dispatching idle periods, start-of-service epochs form a renewal process.

By Eq. (77),

\[ \sum_{i=0}^{\infty} \Pi_i(x) = \Pi_0 + \lim_{x \to \infty} \sum_{i=1}^{\infty} \Pi_i(x) e^x \]

\[ = \Pi_0 + \frac{(1-\rho) \lambda \eta (1-x)}{\eta(\lambda-\lambda x)} \int_0^\infty e^{-\lambda x} [1-H(3)] dx \]

\[ = \Pi_0 + \frac{(1-\rho) \lambda \eta (1-x)}{\eta(\lambda-\lambda x)} \left( \frac{1}{\lambda - \lambda x} - \int_0^\infty e^{-\lambda x} H(3) dx \right) \]

\[ = \Pi_0 + \frac{(1-\rho) \lambda \eta (1-x)}{\eta(\lambda-\lambda x)} \left( 1 + \int_0^\infty H(3) dx e^{-\lambda x} \right) \]

\[ = \Pi_0 + \frac{(1-\rho) \lambda \eta (1-x)}{\eta(\lambda-\lambda x)} \left( 1 - \int_0^\infty e^{-\lambda x} H(3) dx \right) \]

\[ = \Pi_0 + \frac{(1-\rho) \lambda \eta (1-x)}{\eta(\lambda-\lambda x)} \left( 1 - \eta \lambda \rho dx \right). \]

Finally, substitution of \( \Pi_0 = 1-\rho \), by (6), and simplification give as the result:
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\[
\sum_{j=0}^{\infty} P_j \lambda^j = \frac{1}{\eta(\lambda - \eta)} (1 - \eta).
\]  

Equation (9) is seen to be identical to (8.10) which gives the probability generating function of the departure state. This is the direct consequence of the equality \( P_0 = \Pi_0 \).

Let \( W_j(x) \) be the conditional waiting time distribution function, given \( N = j \), and let \( W(x) \) be the unconditional waiting time distribution function, assuming service in order of arrival. Then

\[
W(x) = \sum_{j=0}^{\infty} P_j W_j(x) = P_0 + \sum_{j=1}^{\infty} P_j W_j(x) = P_0 + \sum_{j=1}^{\infty} P_j \left( \frac{\sigma(R_k - 3)N_j}{H \eta - 9} \right)
\]

where, we have used that \( W_j(x) \) is the convolution of the distribution of the remaining service time, given \( N = j \), and the distribution of a sum of \( j - 1 \) independent service times. By definition of \( P_j(x) \), then,

\[
W(x) = P_0 + \sum_{j=1}^{\infty} P_j \left( x - \sigma(3) \right) dH \eta^j \Xi_j(x).
\]

Hence,

\[
\omega(s) = \int_0^{\infty} e^{-sx} dW(x) = P_0 + \sum_{j=1}^{\infty} \left( \int_0^{\infty} e^{-sx} dW_j(x) \right) = P_0 + \sum_{j=1}^{\infty} P_j \omega_j(s) \eta(s - \eta(j))
\]

Thus, the Laplace-Stieltjes transform of the waiting time is

\[
\omega(s) = \frac{1}{\eta(s - \eta(1 - \eta))},
\]

which, of course, is the same as Eq. (8.38).
Let $T_i$ and $T_j$ be an arbitrary pair of consecutive annual epochs, and let the associated arrival states be $(N_i, R_i)$ and $(N_j, R_j)$, respectively. Eqs. (12), (13) and (14) are derived on the assumption of identical state probability distributions at $T_i$ and $T_j$. The distribution at $T_i$, defined by $P_i = P(N_i = 0)$, $P_{i1}(x) = P(N_i = 1, R_i \leq x)$ and $P_{i2}(x) = P(N_i = 2, R_i \leq x)$ for $i = 2, 3, \ldots$ is found by conditioning on $(N_i, R_i)$. The equations reflect the fact that $N_i = x$ may occur if and only if $N_i = x + 1$.

$$T_0 = T_0 \int_0^\infty e^{-\lambda t} d\lambda(t) + \sum_{k=1}^\infty \int_0^\infty e^{-\lambda y} d\lambda(t) \int_0^\infty e^{-\lambda \eta} d\lambda \eta^k(y) .$$  

Eq. (12) is obtained by rewriting $P(N_i = 0) = P(N_i = 0) + \sum_{k=1}^\infty e^{-\lambda y} d\lambda(t) \int_0^\infty e^{-\lambda \eta} d\lambda \eta^k(y)$ and, observing that $P(N_i = 0 | N_i = 0) = 1 e^{-\lambda x} d\lambda(t)$ and $P(N_i = 0 | N_i = k, R_i = x) = 1 e^{-\lambda x} d\lambda(t)$.

$$T_i(x) = T_0 \int_0^\infty [H(t+x) - H(t)] \lambda e^{-\lambda t} dt$$

$$+ \sum_{k=1}^\infty \int_0^\infty e^{-\lambda y} d\lambda(t) \int_0^\infty e^{-\lambda \eta} d\lambda \eta^k(y) \int_0^\infty [H(t+x) - H(t)] \lambda e^{-\lambda t} dt .$$

Eq. (13) follows from $P(N_i = 1, R_i \leq x) = P(N_i = 0) + P(N_i = 1, R_i \leq x)$, rewritten $T_i(x) = T_0 \sum_{k=1}^\infty e^{-\lambda y} [H(t+x) - H(t)] \lambda e^{-\lambda t} dt$ and $P(N_i = 0 | N_i = x) = 1 e^{-\lambda x} d\lambda(t)$ and $P(N_i = 0 | N_i = k, R_i = x) = 1 e^{-\lambda x} d\lambda(t)$.

$$T_j(x) = \int_0^\infty [T_j(t+x) - T_j(t)] \lambda e^{-\lambda t} dt$$

$$+ \sum_{k=1}^\infty \int_0^\infty e^{-\lambda y} d\lambda(t) \int_0^\infty e^{-\lambda \eta} d\lambda \eta^k(y) \int_0^\infty [H(t+x) - H(t)] \lambda e^{-\lambda t} dt .$$

Eq. (14) may be derived by the same kind of arguments that were used for Eqs. (12) and (13).

The double integral of Eq. (12) equals $\int_0^\infty e^{-\lambda x} d\lambda(t) \int_0^\infty e^{-\lambda \eta} d\lambda \eta^k(y)$. Hence, using the Laplace-Stieltjes transform definitions, Eq. (12) becomes

$$T_0 = T_0 \eta (\lambda) + \sum_{k=1}^\infty \eta^k(\lambda) \eta(\lambda) .$$
Eqs. (13) may be written

\[ \tau_1(x) = \kappa_1(x) \left[ \tau_0 + \sum_{n=1}^{\infty} \psi_n(x) \eta_n(x) \right] \]

where

\[ \kappa_1(x) = \int_0^x \left[ H(t-x) - H(t) \right] \lambda e^{-\lambda t} dt. \]

Hence,

\[ \psi_1(x) = \int_0^x e^{-\lambda t} \tau_1(x) dt = -\int_0^x e^{-\lambda t} \kappa_1(x) \sum_{n=1}^{\infty} \psi_n(x) \eta_n(x) dt. \]

Now,

\[ \int_0^x e^{-\lambda t} \kappa_1(x) dt = \int_0^x \int_0^x e^{-\lambda t} \kappa_1(x) e^{-\lambda (t-x)} dt \]

\[ = \lambda \int_0^x e^{-\lambda t} \left( \int_0^x e^{-\lambda (t-x)} dt \right) dt = \lambda \int_0^x e^{-\lambda t} dt = \lambda \int_0^x e^{-\lambda t} \left( 1 - e^{-\lambda x} \right) dt, \]

so that

\[ \int_0^x e^{-\lambda t} \kappa_1(x) dt = \frac{\lambda}{\alpha - \lambda} \left( \eta \right) = \frac{\lambda}{\alpha - \lambda} \left( \eta \right). \]

Hence,

\[ \psi_1(x) = \frac{\lambda}{\alpha - \lambda} \left[ \eta(x) - \eta_0 \right] \tau_0 + \sum_{n=1}^{\infty} \psi_n(x) \eta_n(x). \] (16)

Eq. (14) may be written

\[ \tau_1(x) = \kappa_1(x) + \sum_{n=1}^{\infty} \psi_n(x) \eta_n(x) \] (j = 1, 2, ...) \]

where \( \kappa_1(x) \) has been defined above and

\[ \kappa_1(x) = \int_0^x \left[ H(t-x) - H(t) \right] \lambda e^{-\lambda t} dt \] (j = 1, 2, ...).

Hence,

\[ \psi_1(x) = \int_0^x e^{-\lambda t} \tau_1(x) dt = \int_0^x e^{-\lambda t} \kappa_1(x) dt + \sum_{n=1}^{\infty} \psi_n(x) \eta_n(x) \] (j = 1, 2, ...).

Proceeding precisely as when \( \int_0^x e^{-\lambda t} \kappa_1(x) dt \) was calculated, we derive

\[ \int_0^x e^{-\lambda t} \kappa_1(x) dt = \frac{\lambda}{\alpha - \lambda} \left[ \psi_{j-1}(x) - \psi_1(x) \right] \] (j = 2, 3, ...).

Hence, for \( j = 1, 2, \ldots \),

\[ \psi_j(x) = \frac{\lambda}{\alpha - \lambda} \left[ \psi_{j-1}(x) - \psi_1(x) \right] + \frac{\lambda}{\alpha - \lambda} \left[ \eta(x) - \eta_0 \right] \sum_{n=1}^{\infty} \psi_n(x) \eta_n(x). \] (17)
(Chap. 5, Ex. 14.9)

2. Substitution of (16) and (17) into (4) leads to
\[ u(s, x) = \frac{\lambda}{s - \lambda} \sum_{i=1}^{\infty} \left[ \psi_i(s) - \psi_i(x) \right] s^i + \frac{\lambda}{s - \lambda} \left[ \eta(s) - \eta(x) \right] \left[ \Pi_0 + \frac{\psi(s) + \psi(x)}{\eta(s) - \eta(x)} \right] \]

Now,
\[ \sum_{i=1}^{\infty} \psi_i(s) - \psi_i(x) = \frac{u(s, x) - u(x, s)}{s - x} \]

and
\[ \sum_{i=1}^{\infty} \psi_i(s) \eta^*(x) s^i = \sum_{k=1}^{\infty} \psi_k(s) \eta^*(x) \sum_{i=1}^{k} \left( \frac{s}{\eta(s)} \right)^i \]
\[ - \frac{x}{\eta(s) - \eta(x)} \sum_{k=1}^{\infty} \psi_k(s) \eta^*(x) \left[ 1 - \left( \frac{s}{\eta(s)} \right)^k \right] \]
\[ = \frac{x}{\eta(s) - \eta(x)} \left[ u(s, \eta(x)) - u(x, s) \right] \]

It follows easily that
\[ u(s, \lambda) = x u(s, \lambda) - x u(x, \lambda) + \lambda x \left[ \eta(s) - \eta(x) \right] \left[ \Pi_0 + \frac{u(s) + u(x)}{\eta(s) - \eta(x)} \right] \]  
(18)

3. For \( s = \lambda - \lambda \alpha \), Eq. (18) specializes to
\[ 0 = \lambda x u(x, \lambda) + \lambda x \left[ \eta(s) - \eta(x) \right] \left[ \Pi_0 + \frac{u(s) + u(x)}{\eta(s) - \eta(x)} \right] \]

whereby
\[ \Pi_0 = \frac{u(s, \lambda) - u(x, \lambda)}{\eta(s) - \eta(x)} \] \( \eta(s) - \eta(x) \)  
(19)

Substitution of this expression into (18) and solution w.r.t. \( u(\lambda, x) \) gives
\[ u(s, x) = \frac{x \left[ \eta(s) - \eta(x) \right] u(s, \lambda)}{\eta(s) - \eta(x) \eta(s - \lambda) - \eta(x - \lambda) \eta(x)} \]  
(20)

4. By (15),
\[ u(\lambda, \eta(x)) = \Pi_0 \left[ \eta(x) - \eta(x - \lambda) \right] \]  
(21)

Substitution of this expression into (19) and solution w.r.t. \( u(\lambda, x) \) gives
\[ u(s, x) = \frac{\Pi_0 \left[ \eta(x) - \eta(x - \lambda) \right]}{x - x \eta(x - \lambda) - \eta(x - \lambda) \eta(x)} \]  
(22)
Finally, substitution of (11) into (20) yields
\[
u(\eta, z) = \frac{\Pi(1-z) \eta(1-z)}{\eta(\omega) - \eta(z)}.
\] (5)

We shall show that inversion of (5) gives
\[
\int_{\frac{1}{2}}^{\infty} \Pi_{\delta}(x) x^\gamma = A(\alpha) \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} \left[ H(\gamma x) - H(\gamma) \right] d\gamma,
\] (7)

where
\[
A(\alpha) = \frac{(1-\varphi) \lambda \eta(1-z)}{\eta(\omega) - \eta(z)}
\]

To begin, we show that the Laplace-Stieltjes transform of the LHS of (7) equals the LHS of (5):
\[
\int_{\frac{1}{2}}^{\infty} e^{-\alpha y} \left( \sum_{\delta=1}^{\infty} \Pi_{\delta}(x) y^\delta \right) = \sum_{\delta=1}^{\infty} \Pi_{\delta}(x) y^\delta = \sum_{\delta=1}^{\infty} \psi_\delta(\omega) y^\delta = \nu(\eta, \omega).
\]

Next we show that the Laplace-Stieltjes transform of the RHS of (7) equals the RHS of (5):
\[
\int_{\frac{1}{2}}^{\infty} e^{-\alpha y} A(\alpha) \left[ e^{-\alpha(1-x)} \left( H(\gamma x) - H(\gamma) \right) \right] d\gamma = A(\alpha) \int_{\frac{1}{2}}^{\infty} \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} dH(\gamma x) \right) dx
\]
\[
= \frac{A(\alpha)}{\lambda \eta(z)} \int_{\frac{1}{2}}^{\infty} \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} dH(\gamma) \right) dx
\]
\[
= A(\alpha) \frac{e^{-\alpha(1-z)} \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} dH(\gamma) \right) dx}{\lambda \eta(z)}
\]
\[
= A(\alpha) \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-z)} \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} dH(\gamma) \right) dx
\]
\[
= A(\alpha) \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-z)} \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} dH(\gamma) \right) dx
\]
\[
= A(\alpha) \frac{e^{-\alpha(1-z)} \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} dH(\gamma) \right) dx}{\lambda \eta(z)}
\]
\[
= A(\alpha) \frac{e^{-\alpha(1-z)} \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} dH(\gamma) \right) dx}{\lambda \eta(z)}
\]
\[
= A(\alpha) \frac{e^{-\alpha(1-z)} \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} dH(\gamma) \right) dx}{\lambda \eta(z)}
\]
\[
= A(\alpha) \frac{e^{-\alpha(1-z)} \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} dH(\gamma) \right) dx}{\lambda \eta(z)}
\]

Thus, considering the definition of $A(\alpha)$ and the fact, by (6), that $1-\varphi = \Pi_{\delta}$,
\[
\int_{\frac{1}{2}}^{\infty} e^{-\alpha y} A(\alpha) \left( \int_{\frac{1}{2}}^{\infty} e^{-\alpha(1-x)} \left( H(\gamma x) - H(\gamma) \right) d\gamma \right) = \frac{\lambda \eta(z)}{\eta(\omega) - \eta(z)} \frac{\eta(\omega) - \eta(z)}{\lambda \eta(z)}
\]
Chapter 5, Exercise 15

The integro-differential equation of Takacs.

Let $V_t$ be the virtual waiting time in the M/G/1 queue with order-of-arrival service, and define the distribution function $V(t,x)=P(V_t \leq x)$.

(a) Let $h > 0$, and let $K$ be the number of arrivals in the time interval $(t, t+h)$. We shall show that

$$P(V_{t+h} \leq x, K=0) = (1-\lambda h)V(t,x+h) + o(h), \quad (i)$$

$$P(V_{t+h} \leq x, K=1) = \lambda h \int_0^{x-h} H(x+h-y) dV(t,y) + o(h), \quad (ii)$$

whereby as $P(V_{t+h} \leq x, K \geq 2) = o(h)$ and $V(t+h,x) = \frac{\partial}{\partial h} P(V_{t+h} \leq x, K=1), \quad (\ast)$

$$V(t+h,x) = (1-\lambda h)V(t,x+h) + \lambda h \int_0^{x-h} H(x+h-y) dV(t,y) + o(h). \quad (\ast\ast)$$

Eq. (i). Obviously, $P(V_t \leq x | K=0) \equiv P(V_{t+h} \leq x | K=0)$. Consequently, $P(V_{t+h} \leq x | K=0) = P(V_{t+h} \leq x+h | K=0)$. Also, since $V_t$ and $K$ are independent, $P(V_{t+h} \leq x+h | K=0) = P(V_t \leq x+h)$. Thus, $P(V_{t+h} \leq x | K=0) = P(V_{t+h} \leq x+h)$ and,

$$P(V_{t+h} \leq x, K=0) = P(K=0)P(V_{t+h} \leq x) + o(h)P(V_{t+h} \leq x+h)$$

whereby (i) follows.

Eq. (ii). For $K=1$, denote by $t^*$ the time of arrival and by $z$ the service time of the customer arriving in $(t, t+h)$. Observe that $P(V_{t+h} \leq x | K=1) \equiv P(V_{t+h+z} \leq x | K=1)$. Hence, for all $x \geq 0$,

$$P(V_{t+h} \leq x | K=1) = P(V_{t+z} \leq x+h | K=1) - P(V_{t+z} \leq h+x+h | K=1)$$

Since $V_t, z$ and $K$ are independent variables,

$$P(V_{t+z} \leq h+x+h | K=1) = P(V_{t+z} \leq h+x) \int_0^{x+h} H(x-h-y) dV(t,y) + o(h).$$
(Chap. 5, Ex. 15.a)

For the sake of brevity, define

$$F(x, h) = \mathbb{P}(V_t \leq x + h, t+h-t^* < 2-Y|K = 1).$$

Clearly, $t^*$ is uniformly distributed on $[t, t+h]$, independently of $V_t$ and $z$. Using this fact we derive

$$F(x, h) = \int_0^h \int_{t^*}^{t+h} \int_{z}^{z+h} \mathbb{P}(V_t \leq x + y - z, dH(a), dy, dt) \, dH(a) \, dy \, dt$$

$$\leq \int_0^h \int_{t^*}^{t+h} \int_{z}^{z+h} \mathbb{P}(V_t \leq x + y - z, dH(a), dy, dt) \, dH(a) \, dy$$

$$= \mathbb{V}(t, h)[H(x+h) - H(x)].$$

Hence,

$$\lim_{h \to 0} F(x, h) = 0 \quad (x \geq 0).$$

Evidently,

$$P(V_{x+h} \leq x, K = 1) = P(V_{x+h} \leq x | K = 1) = P(V_{x+h} \leq x | K = 1) = P(V_{x+h} \leq x | K = 1).$$

Hence, by previous results,

$$P(V_{x+h} \leq x, K = 1) = (x + o(h))[\int_0^{x+h} H(x+y) dy \mathbb{P}(V_t \leq y) - F(x, h)]$$

$$= x h \int_0^{x+h} H(x+y) dy \mathbb{P}(V_t \leq y) + o(h).$$

This concludes the proof of (ii). The proof of Eq. (9) is complete.

(b) Subtracting $V(t, x)$ on both sides of (9), dividing through by $h$, and letting $h = 0$, we obtain

$$\frac{\delta V(t, x)}{\delta t} = \frac{\delta V(t, x)}{\delta x} - \mathbb{V}(t, x) + \lambda \int_0^{x} \mathbb{H}(x-y) dy \mathbb{P}(V_t \leq y),$$

which is the integro-differential equation of Takács.
(Chap. 5, Ex. 15 c)

**c)** Assuming \( \lim_{\tau \to \infty} \frac{dV(t, x)}{dt} = 0 \) and \( \lim_{\tau \to \infty} V(t, x) = V(x) \), Eq. (1) becomes

\[
\frac{dV(x)}{dx} = V(x) - \int_0^\infty [H(x-y) - V(y)] \, dy \quad (x \geq 0).
\]  

(2)

Now, define the Laplace–Stieltjes transform

\[ \theta(s) = \int_0^\infty e^{-sx} \, dV(x). \]

All three terms in Eq. (2) are functions of \( x \). The L–S transforms of LHS and RHS of the equation are, respectively,

\[
\int_0^\infty e^{-sx} \frac{dV(x)}{dx} = V(0) + \int_0^\infty e^{-sx} \, dV(x)
\]

\[
= V(0) + \int_0^\infty e^{-sx} \, dV(x) + \int_0^\infty e^{-sx} \, [H(x-y) - V(y)]
\]

\[
= V(0) - V(0) + \int_0^\infty e^{-sx} \, [H(x-y) - V(y)]
\]

\[
= \int_0^\infty e^{-sx} \, d\theta(s).
\]

and

\[
\int_0^\infty e^{-sx} \, [H(x-y) - V(y)] = \int_0^\infty e^{-sx} \, V(x) - \int_0^\infty e^{-sx} \, [H(x-y) - V(y)]
\]

\[
= \int_0^\infty e^{-sx} \, d\theta(s).
\]

Since \( \int_0^\infty H(x-y) \, dV(y) \) is the convolution of distribution functions with L–S transforms \( \eta(s) \) and \( \theta(s) \).

Equating LHS and RHS transforms we obtain

\[
\theta(s) - V(0) = \frac{\lambda}{\beta} \theta(s) - \frac{\lambda}{\beta} \eta(s) \theta(s).
\]  

(3)

**d)** Solving (3) for \( \theta(s) \) gives

\[ \theta(s) = \frac{V(0)}{\frac{\lambda}{\beta} - \lambda \eta(s)}. \]

Inserting \( V(0) = 1 - \rho \) as usual for the \( M/G/1 \) queue, we get

(8.38), as we should.
Chapter 5, Exercise 16

Verify that (8.72) is the solution of (8.71).

\[
\begin{align*}
\phi_n^y &= \begin{cases} 
\frac{\gamma}{\beta} e^{-\alpha y} & (n = 1), \\
\frac{\gamma}{\beta} \sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k \alpha^{-k} \phi_{n-k}^y & (n \geq 2). 
\end{cases} \\
\phi_n^y &= \frac{\gamma}{\beta} \sum_{n=1}^{\infty} \frac{(\lambda \alpha n)^{n-1}}{(n-1)!} e^{-\lambda \alpha n} \quad (n \geq 2). 
\end{align*}
\]  

(8.70)  
(8.72)

The proof is by induction. First, we observe that for all feasible \( n \) and \( j \) such that \( n - j > 0 \), Eq. (8.72) reduces to \( e^{\lambda \alpha n} \), which agrees with (8.71). Next, we assume that (8.72) already has been proved for all \( n \) and \( j \) such that \( n - j < k_0 \). We shall show that then (8.72) will hold for all \( n \) and \( j \) such that \( n - j \geq k_0 \).

Thus, assume values of \( n \) and \( j \) such that \( n - j = k_0 \).

By the induction hypothesis, (8.72) applies to all the factors \( \phi_{n-1}, \phi_{n-1-j}, \ldots, \phi_{n-j} \) of (8.71), since \( n - j \leq k_0 \). Substitution of (8.72) into (8.71) and straightforward reduction produce

\[
\begin{align*}
\phi_n^y &= \frac{\gamma}{\beta} \sum_{k=1}^{\infty} \frac{(\lambda \alpha k)^{k-1}}{(k-1)!} e^{-\lambda \alpha k} \\
&= \frac{\gamma}{\beta} \sum_{n=1}^{\infty} \frac{(\lambda \alpha n)^{n-1}}{(n-1)!} e^{-\lambda \alpha n} \quad (n \geq 2).
\end{align*}
\]

By induction, we conclude that (8.72) holds for all \( n \) and \( j \) where \( j \geq 1 \), \( n \geq j \).
Chapter 5, Exercise 17

a. Let \( N_k \) be the number of customers served during a \( k \)-busy period.

b. Assume an M/G/1 queue. Starting at \( t_0 \) with \( i \) customers in the system, we imagine that first we serve the \( i \) customers plus all later arrivals until the moment \( t_1 \) when a departure leaves the original \( i \) customers behind. Clearly, \( [t_0, t_1) \) is an \( i \)-busy period. Let \( N_i \) be the number of customers served during \( [t_0, t_1) \).

Now, serve the remaining \( j \) customers and all later arrivals until, at \( t_2 \), the system is empty. Again, \( [t_1, t_2) \) is a \( j \)-busy period, and we let \( N_j \) be the number of customers served during \( [t_1, t_2) \).

By the independence of interarrival times and service times, as well as the assumption of Poisson arrivals, the realizations of the \( i \)-busy period and the \( j \)-busy period are independent. In particular, \( N_i \) and \( N_j \) are independent, and

\[
N_{i+j} = N_i + N_j
\]

where \( N_{i+j} \) is the number of customers served during the entire \((i+j)\)-busy period.

d. Extending the arguments behind (i), a \( k \)-busy period may be decomposed into \( k \) independent \( 1 \)-busy periods with associated number of services \( N_i(w) \), \( i = 1, 2, \ldots, k \), so that \( N_k = \sum_{i=1}^{k} N_i(w) \). Hence,

\[
E(N_k) = \sum_{i=1}^{k} E(N_i(w)) = k \cdot E(N_1(w))
\]

\(E(N_i)\) is most easily derived from the mean busy period \( b \) as follows: The mean cycle time equals \( b + \lambda^{-1} \). Thus, the average number of busy periods (or cycles) per unit time is \( \frac{b + \lambda^{-1}}{\lambda} \). Since the number of arrivals per unit time (number of services per unit time) is \( \lambda \), the average number of services per busy period will be \( E(N_i) = \frac{\lambda}{b + \lambda^{-1}} = \frac{1}{1-\lambda + \lambda b} \).

According to (8.6a), \( b = \tau / (1-\lambda \tau) \). Hence, \( E(N_i) = 1 / (1-\lambda) = 1 / (1-\phi) \).

It follows that, for \( \phi < 1 \),

\[
E(N_k) = \frac{k}{1-\phi}.
\]
Let $p_i^{(n)}$ denote the probability that $n$ customers will be served during an $i$-bus period. By part a,
\[
\sum_{k=1}^{n-1} p_i^{(n)} f_{m,n} = p_i^{(n)} \quad (i \geq j, n \geq 1).
\]

For an M/D/1 queue, $f_{m,n}$ is given by (8.72), whose substitution into the above equation gives
\[
\sum_{k=1}^{n-1} \frac{m^k}{k!} \left(\frac{m}{n-k}\right)^{n-k} \frac{1}{(n-k)!} = \frac{1}{n} \left(\frac{m}{n-j}\right)^{n-j} \frac{1}{(n-j)!}.
\]

After cancellation of powers of $m$ and a rearrangement, we have
\[
\sum_{k=j}^{n-1} \frac{n^k}{k!} \left(\frac{n}{k-j}\right)^{k-j} \frac{1}{(k-j)!} = \frac{n}{i} \left(\frac{n}{i-j}\right)^{n-j} \frac{1}{(n-j)!} \quad (i \geq 1, j, n \geq 1).
\]

For $i = j = 1$, (e) becomes the identity
\[
\sum_{k=0}^{n-1} \frac{n^k}{k!} \left(\frac{n}{k-1}\right)^{k-1} \frac{1}{(k-1)!} = \frac{n}{n-1} n^{n-1} \quad (n \geq 2).
\]

Chapter 5, Exercise 18

'Let $N_i$ be the number of customers served during an $i$-bus period.'

In the present case, the M/M/1 queue, $H^{(n)}(t) = \frac{m^n}{n^m} e^{-n t}$, so that
\[
\frac{d}{dt} H^{(n)}(t) = \frac{n^m}{(n-1)!} t^{n-1} e^{-n t}.
\]

Setting $t = \infty$ in (8.73) and replacing $dH^{(n)}(t)$ by $H^{(n)}(0^+)$, we find
\[
\mathbb{P}(N_1 = n) = \frac{1}{n} \frac{m^n}{(n-1)!} e^{-n t} \left(\frac{m}{n-1}\right)^{n-1} e^{-m t} \left(\frac{m}{n}\right)^m e^{-m t}.
\]

Since $\int_0^\infty x^a e^{-bx} dx = \alpha^m/\alpha^m$ for $a > b$, $m = 0, 1, 2, \ldots$ (see a Table of Integrals),
\[
\mathbb{P}(N_1 = n) = \frac{m^n}{n^m} e^{-n t} \left(\frac{m}{n}\right)^m e^{-m t}.
\]

so that
\[
\mathbb{P}(N_1 = n) = \frac{1}{n} \left(\frac{m^{n-j}}{(n-j)^{n-j}} \right) \quad (n \geq j),
\]

where $\rho = \lambda/\mu$. \qed
Chapter 5, Exercise 19

The "polite" customer

Throughout in parts a-d an M/G/1 queue is assumed. By a polite customer is meant a customer who declines to enter service when any other customer is present in the queue. Its equilibrium waiting time is denoted by \( W_p \).

a) By definition, a polite customer who arrives while the server is busy will not enter service until the very end of the busy period that would have been realized without the appearance of the polite customer.

Given Poisson arrivals for all customers, we assume that also the polite customer will arrive at a random time in equilibrium. In particular, in case he arrives while the server is busy, then the arrival takes place at a randomly selected point in time in the renewal process of busy periods, disregarding the idle periods. Accordingly, the waiting time is a residual busy period, whose distribution in equilibrium is given by Eq. (79). It follows that

\[
P(W_p > x | W_p > 0) = \frac{1}{b} [1 - B(x/b)],
\]

where \( B(x) \) is the distribution function of the busy period, and \( b \) is its mean.

b) With probability 1 - \( p \) the polite customer arrives at an idle server and has waiting time 0. With probability \( p \) he arrives at a busy server and his conditional waiting time distribution is given by (1). Using that \( b = \tau r/(1-p) \), by Eq. (8.64), we conclude that

\[
P(W_p > x) = (1-p) + \frac{\alpha(1-p)}{\tau} [1 - B(x/b)].
\]

c) If the polite customer has to wait, his waiting time distribution is, by part a, identical to the residual busy period distribution in equilibrium, and so the mean wait equals the mean of the residual busy period given by (7.13). Thus,
(Chap. 5, Ex. 19-e)

\[ E(W_p | W_p > 0) = \frac{b}{2} + \frac{\sigma_b^2}{2b}, \]

where \(\sigma_b^2\) is the variance of the busy period. As \(b = \tau/(1-p)\),

\[ E(W_p | W_p > 0) = \frac{\tau}{2(1-p)} + \frac{\sigma_b^2}{2\tau}. \tag{3} \]

4 By (8.70), \(E(B^2) = \eta(0)/(1-p)^2\). Now, \(\eta(0)\) equals the second moment of the service time distribution, so we may write \(\eta(0) = \sigma^2 + \tau^2\), where \(\sigma^2\) is the variance of the service time. Hence,

\[ \sigma_b^2 = E(B^2) - b^2 = \frac{\sigma^2 + \tau^2}{(1-p)^2} - \frac{\tau^2}{(1-p)^2} = \frac{\sigma^2}{(1-p)^2}. \]

Substitution of this expression for \(\sigma_b^2\) into (3) yields

\[ E(W_p | W_p > 0) = \frac{\tau^2/\sigma^2}{2\tau(1-p)^2}. \tag{4} \]

5 Suppose the polite customer makes his arrival in an M/M/s queue, with arrival rate \(\lambda\) and mean service time \(\mu^{-1}\). There he will wait only if all \(s\) servers are busy at arrival.

The key observation is that all servers' busy periods follow precisely the same distribution as does the busy period in the M/M/1 queue, with arrival rate \(\lambda\) and mean service time \((\mu\lambda)^{-1}\).

We can conclude that \(E(W_p | W_p > 0)\) may be derived by the use of Eq. (9), getting \(\tau = (\mu\lambda)^{-1}\), \(\sigma^2 = \tau^2 = (\mu\lambda)^{-2}\), and \(p = \lambda/\mu\). Hence, for the M/M/s queue,

\[ E(W_p | W_p > 0) = \frac{(\mu\lambda)^2 + (\mu\lambda)^{-2}}{2(\mu\lambda)^{-2}(1-p)^2}, \]

simplifying to

\[ E(W_p | W_p > 0) = \frac{1}{(1-p)^2\mu}. \tag{5} \]
Chapter 5, Exercise 20

Service in reverse order of arrival

a) We consider an arbitrary customer arriving at an M/G/1 queue at a time $T_c$ when the server is busy. Disregarding idle periods, the arrival epoch is a randomly selected point in time in the renewal process where interevent times have probability distribution $H(t)$. Hence, the remaining service time $T - T_c$ has the probability distribution function $H(t)$ given by (9.4), by application of (7.9).

If $T - T_c = t$, then the number of new arrivals during $[T_c, T]$ will follow the Poisson distribution with mean $t$. Thus, the joint probability of $T - T_c = x$ and $j$ new arrivals is

$$P_j(x) = \int_0^x \frac{e^{-\lambda y} \lambda^y}{y!} dy.$$

b) Let, as usual, $W$ be the waiting time of an arbitrary customer (the test customer) and let $W(t)$ be the equilibrium waiting time distribution function, given service in reverse order of arrival. Denoting the arrival state by $N$, clearly

$$W(t) = P(W > t) = P(N = 0)P(W > t | N = 0) + P(N = 1)P(W > t | N = 1).$$

Given Poisson arrivals, $P(N = 1) = \rho$. Hence,

$$W(t) = (1 - \rho) \cdot \rho.\ P(W > t | N = 1).$$

To find $P(W > t | N = 1)$, observe that the waiting time is the sum of the remaining service time $T - T_c$ and a $j$-busy period, where $j$ is the number of arrivals during $[T_c, T]$, since the test customer must wait until both these $j$ customers and all later arrivals have been served.

Note that for given $j$, the conditional remaining service time and the $j$-busy period are independent. It follows that, given $N = 1$, the joint probability of $j$ new arrivals and a total wait of less than $t$ equals

$$\int_0^t P_j(t - x) dB_j(x),$$

where $B_j(x)$ is the distribution function of the $j$-busy period. Hence,
(Chap. 5, Ex. 20 b)

so that

\[ W(t) = (1-\rho) + \rho \sum_{i=0}^{\infty} \int_{0}^{\infty} F(t-x) d B_i(x). \] (1)

2. Let \( \omega(s) \) denote the Laplace–Stieltjes transform of the waiting time distribution function \( W(t) \), and let, as before, \( \eta(s) \) and \( \rho(s) \) be the Laplace–Stieltjes transforms of service time and busy period distribution functions, respectively.

By (1), and part 3,

\[ \omega(s) = \int_{0}^{\infty} e^{-st} dW(t) \]

\[ = (1-\rho) + \rho \sum_{i=0}^{\infty} \left( \int_{0}^{\infty} e^{-st} d F_t(x) \right) \left( \int_{0}^{\infty} e^{-st} d B_i(x) \right) \]

\[ = (1-\rho) + \rho \sum_{i=0}^{\infty} \left( \int_{0}^{\infty} e^{-st} \left( \frac{\lambda}{s} \right) e^{-\lambda t} d B_i(x) \right) \beta_i(s) \]

\[ = (1-\rho) + \rho \sum_{i=0}^{\infty} \left( \int_{0}^{\infty} e^{-\lambda t} \left( \frac{\lambda}{s} \right) \beta_i(s) \right) d H(x) \]

\[ = (1-\rho) + \rho \sum_{i=0}^{\infty} \left( \frac{\lambda}{s} \right) \beta_i(s) \int_{0}^{\infty} e^{-\lambda t} d H(x) \]

The last integral is the Laplace–Stieltjes transform of the remaining service time distribution function, evaluated at \( s + \lambda E[\rho] \). According to Eq. (7.10) this transform equals

\[ \int_{0}^{\infty} e^{-\lambda x} d H(x) = \frac{1}{\lambda} \frac{1-\eta(s)}{s}, \]

where \( \tau \) is the mean service time. Hence, since \( \rho = \lambda \tau \),

\[ \omega(s) = (1-\rho) + \lambda \frac{1-\eta(s+\lambda \tau)}{s+\lambda \tau E[\rho]} \]

3. By (b.67), \( \rho(s) = \eta(s+\lambda \tau - \rho(s)) \), so that (2) reduces to

\[ \omega(s) = (1-\rho) + \frac{\lambda (1-\eta(s))}{s+\lambda E[\rho]} \]

\[ \rho(s) = \frac{\lambda (1-\eta(s))}{s+\lambda E[\rho]} \] (3)
Chapter 5, Exercise 21

It is required to calculate the arriving customer's equilibrium distribution \( \Pi_j \) for the \( M/G/1 \) queue with \( n \) waiting positions.

1. Let \( \Pi_j^* \) (\( j = 0, 1, \ldots, n \)) denote any unnormalized \( \Pi_j^* \) calculated from (9.1) starting with an arbitrary positive value of \( \Pi_0^* \), and set

\[
d = \frac{\Pi_0^*}{\Pi_1^* + \cdots + \Pi_n^*}.
\]

By (9.12) and (9.13),

\[
\Pi_j = \frac{\Pi_j^*}{\Pi_0^* + d} = \frac{\Pi_j^* d}{\Pi_0^* d + ad} = \frac{\Pi_j^*}{\Pi_0^* + d N_j} \quad (j = 0, 1, \ldots, n),
\]

\[
\Pi_{j+1} = \frac{\Pi_j^* + (N_j - 1) \Pi_{j+1}^*}{\Pi_0^* + d N_j} = \frac{\Pi_j^* + (N_j - 1) \Pi_{j+1}^*}{\Pi_0^* + d N_j}.
\]

2. Suppose that the state distribution \( \Pi_j \) for the corresponding infinite waiting-room queue has been calculated.

By proportionality of \( \{\Pi_j\} \) and \( \{\Pi_j^*\} \) for \( j = 0, 1, \ldots, n \), the starting value \( \Pi_0^* = \Pi_0 \) will lead to \( \Pi_j^* = \Pi_j \) for \( j = 0, 1, \ldots, n \) and

\[
d = \sum_{j=0}^n \Pi_j.
\]

By part a, then,

\[
\Pi_j = \frac{\Pi_j}{\Pi_0 + d \sum_{j=0}^n \Pi_j} \quad (j = 0, 1, \ldots, n),
\]

\[
\Pi_{j+1} = \frac{\Pi_j + (N_j - 1) \sum_{j=0}^n \Pi_j}{\Pi_0 + d \sum_{j=0}^n \Pi_j}.
\]

3. For an \( M/M/1 \) queue \( \Pi_j^* = (1-a)^j \) (\( j = 0, 1, \ldots, n \)), see Ex. 4 of Chapter 1. By substitution into the equations of part b:

\[
\Pi_j = \frac{(1-a)^j}{1-a^{n+1}} \quad (j = 0, 1, \ldots, n+1)
\]

The "rate up - rate down" equations are \( \lambda P_j = \mu P_{j+1} \) (\( j = 0, 1, \ldots, n \)), by which \( \Pi_j = P_j = (1-a)/(1-a^{n+1}) \), where \( a = \lambda/\mu \), in agreement with the above result. \( \square \)
Chapter 5, Exercise 22

'A particle-counting device...

The system may be modeled as a single-server queue with mixing
room of size 2, gross arrival rate \( \lambda \), effective arrival rate in
state \( j \) equal to \( \lambda_j = (3-j) \lambda \), constant service time \( \tau = 1 \). Let

\[ b_j = \text{mean of a } j \text{-busy period } (j = 1, 2), \]

\[ p(j|1) = \text{probability that } i \text{ buffers fill } (i \text{ particles arrive}
\text{ at idle buffers) during a service time, given state } j \text{ } (j = 1, 2) \text{ at the start of service}. \]

Clearly,

\[ b_1 = 1 + p(1|1) b_1 + p(1|1) b_2, \quad (1) \]

\[ b_2 = 1 + p(1|2) b_1 + p(1|2) b_2. \quad (2) \]

Using \( p(2|2) + p(1|2) = 1 \), Eq (2) can be written

\[ b_2 = \frac{1 + p(1|2) b_1}{p(1|2)}. \quad (3) \]

Substitution of (3) into (1) and use of \( p(1|2) + p(1|1) + p(1|1) = 1 \) give

\[ b_1 = \frac{p(1|2) + p(1|1)}{p(1|2) p(1|1)}. \quad (4) \]

The probability that an idle buffer will be filled during a service
period equals \( 1 - e^{-x} \). Hence, \( p(1|2) = e^{-x} \), \( p(1|1) = (1 - e^{-x})^2 \), and
\( p(0|1) = e^{-3x} \). Substitution into (4) and simplification yield

\[ b_1 = e^{-3x} + e^{-x} - e^{-5x}. \quad (5) \]

As \( \lambda = 3 \lambda \), the mean idle period equals \( (3\lambda)^{-1} \). Hence, in analogy
with (4.14), the carried load is given by

\[ \rho = \frac{b_1}{(3\lambda)^{-1} + b_1}. \quad (6) \]

The offered load is

\[ \alpha = (3\lambda) \tau = 3 \lambda. \quad (7) \]

By (5),(6) and (7),

\[ p = 1 - \frac{\rho}{\alpha} = 1 - \frac{e^{-3x} + e^{-x} - e^{-5x}}{3 \lambda (e^{3x} + e^{-x} - e^{-5x})}. \]
Chapter 5, Exercise 23

Let \( B(j,k) \) be the duration of the \( j \)-busy period in the \( M/G/1 \) queue with \( j+k-1 \) waiting positions.

Observe that the system may hold altogether \( j+k \) customers. Let \( EB(j,k) = b(j,k) \), and let \( P(j,k) \) be the probability that throughout the busy period there will always be at least one unoccupied waiting position.

\( a \) Suppose service begins when there are \( j \) customers in the system. We may assume that customers are served in reverse order of arrival. Initially, we decompose the \( j \)-busy period into two independent time intervals. The first interval is the time needed to reduce the state from \( j \) to \( j-1 \). This time interval is distributed as \( B(j,1) \). The second interval is the time needed to reduce the state from \( j-1 \) to 0, so this time interval is distributed as \( B(j-1,k+1) \) assuming \( j \geq 2 \). Hence,

\[
B(j,k) = B(j,1) + B(j-1,k+1) \quad (j \geq 2),
\]

so that

\[
B(j,k) = \sum_{i=k}^{j-1} B(i,1).
\]

\( b \) Taking means in (2) we obtain

\[
b(j,k) = \sum_{i=k}^{j-1} b(i,1).
\]

We now decompose the \( j \)-busy period in a different way. Imagine that the first customer, \( C_1 \), if any, who fills up the queue will not enter service until there are no other waiting customers. Then the time until \( C_1 \), should he exist, will get served is distributed as \( B(j,k-1) \), assuming \( k \geq 1 \). With probability \( 1 - P(j,k) \), 0 will arrive during the busy period and thus generate a \( j \)-busy period distributed as \( B(j,j+k-1) \).

We conclude that

\[
B(j,k) = B(j,k-1) + B(j,j+k-1) \quad (k \geq 1),
\]
where $I = 0$ if $C$ does not arrive and $I = 1$ if he does. Note that $I$ and $B(i,j,k,l)$ are independent variables. Taking means in (4) we derive
\[ b(j,k) = b(j,k-1) + [1 - P(j,k)]b(i,j,k,l) \quad (k \geq 1). \quad (5) \]

2. Writing $b(j,k)$ and $b(j,k-1)$ as sums of mean busy periods, by the use of (5), Eq. (5) yields
\[ P(j,k) = \frac{b(k,j-1)}{b(i,j,k,l)} \quad (k \geq 1). \quad (6) \]

3. Now assume exponential service times with mean $\mu^{-1}$ and let $a = \lambda/\mu$. In this case the mean busy periods are
\[ M/M/I/n : \quad b(l,n) = \frac{1}{\mu} \sum_{i=0}^{n} a^i \quad (n \geq 0). \quad (7) \]
This formula might be derived from (9.8). Alternatively, one can use the relation $b(l,n)/\lambda = (1 - P)/P$ (compare with Eq. (6.10) of Chapter 3), plus the fact that $P = 1/e^{2a}$. By (6) and (7),
\[ M/M/I/j+k-1 : \quad P(j,k) = \frac{\sum_{i=0}^{j+k-1} a^i}{\sum_{j=0}^{\infty} a^i} \quad (k \geq 1). \quad (8) \]

4. Define $P(j,k,p) = P(j,k)$, and let $P(j,k,p)$ denote the gambler's ruin probability in a game where he starts with $j$ units, the adversary starts with $k$ units, and the probability of his winning 1 unit is $p$ in each trial (thus the adversary's winning probability is $q = 1 - p$). We shall show that
\[ P(j,k,p) = P(j,k,p,q). \quad (9) \]

Consider an $M/M/I$ queue with $j+k$ petitions (incl. service) where $j \geq k$, and $i$ is the initial state. In state $i$, $1 \leq i \leq j+k-1$, the transition probabilities are $P(i+1) = \lambda/(i+\nu) = \alpha/(i+\nu) = p, P(i-1) = 1 - p = q$. One sees that the embedded Markov chain (arrivals and departures) of the state variable $i$, exactly simulates the game as described if $\lambda$ and $\mu$ satisfy $\lambda/(\lambda+\mu) = p$. Eq. (9) follows.
'Consider the equilibrium M/G/1 queue with batch arrivals.'

\[ W = \text{time from arrival to start of service of the test customer's batch} \]
\[ W' = \text{remaining time until start of service of the test customer} \]

Let \( \omega_1(s) \) and \( \omega_2(s) \) denote the Laplace-Stieltjes transforms of \( W \) and \( W' \), respectively. As \( W \) and \( W' \) are independent, \( W = W_1 + W_2 \) has Laplace-Stieltjes transform

\[ \tilde{\omega}(s) = \omega_1(s) \omega_2(s) \] (1)

**Derivation of \( \omega_1(s) \)**

Let \( \tau_0 \) be the mean and let \( \eta_0(s) \) be the Laplace-Stieltjes transform of the batch service time. By (3.96),

\[ \omega_1(s) = \frac{s(1-\lambda \tau_0)}{s - \lambda[1 - \eta_0(s)]} \]

By Exercise 5, \( \tau = m \tau_0 \) and \( \eta(s) = g(\eta(s)) \), where \( \tau \) and \( \eta \) are mean and Laplace-Stieltjes transform of the individual service times, and \( m \) and \( g(s) \) are mean and probability generating function of the number of customers in a batch. Thus,

\[ \omega_1(s) = \frac{s(1-\lambda m \tau_0)}{s - \lambda[1 - g(\eta(s))]} \] (2)

**Derivation of \( \omega_2(s) \)**

Let \( N \) and \( N' \) be, respectively, the size of an arbitrary batch and a test customer's batch, and let \( N' = n \) be the number of batch customers served ahead of the test customer. We may assume that the customers in a batch are served in random order, but batches must be served in order of arrival.

Clearly,

\[ P(N' = k) = \sum_{j=0}^{\infty} \frac{1}{j!} P(N' = j) \quad (k = 0, 1, \ldots) \]

Substitution therein of \( P(N' = j) = \frac{1}{j!} P(N' = j) \) in, by (10.6), results in
Denoting by $h(z)$ the probability generating function of $N^*$, we find

$$h(z) = \sum_{k=0}^{\infty} P(N^* = k) z^k$$

$$= \frac{1}{m} \sum_{k=0}^{\infty} \exp(-\mu z) \rho(z^k) z^k$$

$$= \frac{1}{m} \sum_{k=0}^{\infty} \exp(-\mu z) \sum_{i=0}^{k} \frac{z^k}{i!} P(N^* = i) z^k$$

$$= \frac{1}{m} \sum_{k=0}^{\infty} \exp(-\mu z) \sum_{i=0}^{k} \frac{z^i}{i!} P(N^* = i) z^k$$

$$= \frac{1}{m} \sum_{k=0}^{\infty} \exp(-\mu z) \sum_{i=0}^{k} \frac{z^i}{i!} P(N^* = i) z^k$$

$$= \frac{1}{m} \left[ 1 - \sum_{k=0}^{\infty} \rho(z^k) z^k \right] \sum_{k=0}^{\infty} z^k,$$

whereby

$$h(z) = \frac{1 - g(z)}{m \eta(z)}.$$ (3)

Now, $W_1$ is the sum of $N^*$ independent service times, each with Laplace-Stieltjes transform $\eta(s)$. By Exercise 5,

$$\psi_1(s) = h(\eta(s)).$$ (4)

By (3) and (4),

$$\psi_1(s) = \frac{1 - g(s)}{m \eta(s)}.$$ (5)

Finally, combining (1), (2) and (5), we obtain the Laplace-Stieltjes transform of an arbitrary customer's total waiting time $W$,

$$\tilde{\omega}(s) = \frac{1}{s} \frac{1 - g(s)}{m \eta(s)} - \frac{\lambda m r^*}{s - \lambda (1 - g(s)) \eta(s)}.$$

\[\square\]
Chapter 5, Exercise 25

Let \( c \) be the minimum mean operating cost per unit time…

We assume an M/G/1 queue and the choice between continuous operation and some \( N \)-policy. As a consequence of the preceding analysis we distinguish between two cases as follows.

**Case 1:** \( c_0 < \lambda c_0(n^*)/(1-p) \)

It is known already that in this case \( c \) is minimized by continuous operation. Per unit time, the three cost elements are:

- **Running cost** = \( c_0 \)
- **Switching cost** = \( c_s \)
- **Holding cost** = \( \lambda c_0 E(X) \)

Now, \( E(X) = \tau + E(W) \), and the mean waiting time \( E(W) \) is given by the Pollaczek-Khintchine formula (8.39). Thus \( c = c_0 + \lambda c_0 E(X) \) becomes

\[
c = c_0 + c_s \left[ \frac{\sigma^2}{2(1-p)} \left( 1 + \frac{\sigma^2}{\tau} \right) \right] \quad (c_0 < \frac{\lambda c_0(n^*)}{1-p}) \quad (\ast)
\]

**Case 2:** \( c_0 > \lambda c_0(n^*)/(1-p) \)

In this case \( c \) is minimized by choosing an \( N \)-policy with parameter \( n = n^* \). The calculation of the corresponding cost \( c_0(n^*) \), namely the minimal variable cost per customer, has been described previously. The minimal variable cost per unit time is seen to equal \( \lambda c_0(n^*) \); \( c \) is obtained by adding these fixed costs (independent of \( n \)) that were not taken into consideration in calculating \( n^* \).

First, there is a running cost per unit time equal to \( c_0 \phi \), since, for any \( n \), the server will be busy \((n-1)\) the fraction \( \phi \) of the time. Second, there is the holding cost \( \lambda c_0 E(X) \) of a system where the server is on whenever a customer is present. Thus, \( c = c_0 \phi + \lambda c_0(n^*) + \lambda c_0 E(X) \), which becomes

\[
c = c_0 \phi + \lambda c_0(n^*) + c_s \left[ \frac{\sigma^2}{2(1-p)} \left( 1 + \frac{\sigma^2}{\tau} \right) \right] \quad (c_0 > \frac{\lambda c_0(n^*)}{1-p}) \quad (\ast \ast)
\]
(Chap 5, Ex 25)

Examples.

In every case, \( \lambda = 0.75 \), \( \tau = 1 \) (so that \( \lambda \tau = 0.75 \)), \( c = 12 \) and \( c_0 = 2 \).
Hence, \( n^* = 2 \), \( c_0(n^*) = 5 \), and the borderline value for \( c \) is
\[
\theta_c = \lambda c_0(n^*)/(1-\tau) = 5.
\]

(a) \( c_0 = 4 \). As \( c_0 < \theta_c \), \( c \) is minimized by a do-nothing policy.
Thus, Eq. (**) applies. If service times are exponentially distributed,
then \( \sigma^2 = n^2 = 1 \) and, by (**), \( c = 6 \). If service times are constant,
then \( \sigma^2 = 0 \) and, by (**), \( c = 5 \frac{1}{2} \).

(b) \( c_0 = 6 \). As \( c_0 > \theta_c \), \( c \) is minimized by an \( N \)-policy with \( n^* = 2 \).
Thus, Eq. (**) applies. If service times are exponentially distributed,
then \( \sigma^2 = n^2 = 1 \) and, by (**), \( c = 7 \). If service times are constant,
then \( \sigma^2 = 0 \) and, by (**), \( c = 7 \).

Chapter 5, Exercise 26

'Consider the M/G/1 queue operating under a T-policy with parameter t'.

(a) The probability that no customer will arrive during a vacation of
length \( t \) is \( P(0) = e^{-\lambda t} \). Hence, with \( Y \) being the consecutive number
of vacations with no arrivals, \( P(Y = i) = (e^{-\lambda t})^i(1-e^{-\lambda t}) \), whereby
\[
E(Y) = \sum_{i=0}^{\infty} i P(Y = i) = \frac{e^{-\lambda t}}{1-e^{-\lambda t}}.
\]

(b) Given Poisson traffic, the number of arrivals during a vacation
has the Poisson distribution with mean \( \lambda t \). Hence,
\[
f_\lambda(a) = \frac{e^{-a} a^\lambda}{\lambda!} e^{-\lambda t} a^t = e^{-\lambda t} (\frac{a}{\lambda})^\lambda.
\]

(c) \( f_\lambda'(a) = \lambda t e^{-\lambda t} a^t \), \( f_\lambda''(a) = \lambda t e^{-\lambda t} a^t \),
\[
f_\lambda'(1) = \lambda t , \quad f_\lambda''(1) = \lambda t e^t.
\]

If in (1.14) we substitute \( P(0) = e^{-\lambda t} \) and the above expressions
for \( E(Y) \), \( f_\lambda'(1) \) and \( f_\lambda''(1) \), we obtain (1.24).
Chapter 5, Exercise 27

The Maclaurin series method for the M/G/1 random service queue.

\[ f(s) = \mathbb{P}(W > t | W > 0) = 1 - H(s) + \sum_{j=2}^{\infty} \left( \frac{s^{j-2}}{(j-2)!} \right) h^{(j-2)}(0) \Phi_j(0), \quad (12.9) \]

\[ \tilde{W}_j(s) = 1 - H(s) + \sum_{r=0}^{\infty} \frac{s^{j+r-2}}{(j+r-2)!} h^{(j+r-2)}(0) \tilde{W}_{j+r}(s) ds \quad (j = 2, 3, \ldots). \quad (12.10) \]

Assume that \( \tilde{W}_j(s) \) has the Maclaurin series expansion

\[ \tilde{W}_j(s) = \sum_{r=0}^{\infty} \frac{s^r}{r!} \tilde{W}_j^{(r)}(0) \quad (j = 1, 2, \ldots; \tilde{W}_1^{(r)}(0) = 1). \quad (1) \]

Suppose \( H(s) \) is continuous and differentiable, and set

\[ h(x) = \frac{d}{ds} H(s), \quad (2) \]

and let

\[ \tilde{h}_j(x) = h(x) \tilde{W}_j(x). \quad (3) \]

Also, for any function \( f(x) \), define \( f^{(r)}(x) = \left( \int_{0}^{x} \frac{d^r}{ds^r} f(s) \right) ds \).

By (12.10), (2) and (3),

\[ \tilde{W}_j(s) = 1 - H(s) + \sum_{r=0}^{\infty} \frac{s^{j+r-2}}{(j+r-2)!} \left( \int_{0}^{s} \frac{d^{j+r-2}}{ds^{j+r-2}} h^{(j+r-2)}(s) \tilde{W}_{j+r}(s) ds \right) \quad (j = 2, 3, \ldots). \]

Repeated differentiation w.r.t. \( x \) yields

\[ \tilde{W}_j^{(n)}(s) = -H^{(n)}(s) + \sum_{r=0}^{n} \frac{s^{j+r-2}}{(j+r-2)!} \left( \int_{0}^{s} \frac{d^{j+r-2}}{ds^{j+r-2}} h^{(j+r-2)}(s) \tilde{W}_{j+r}(s) ds \right) \]

for \( j = 2, 3, \ldots, n = 1, 2, \ldots \).

Hence,

\[ \tilde{W}_j^{(n)} = -H^{(n)} + \sum_{r=0}^{n} \frac{s^{j+r-2}}{(j+r-2)!} \left( \int_{0}^{s} \frac{d^{j+r-2}}{ds^{j+r-2}} h^{(j+r-2)}(s) \tilde{W}_{j+r}(s) ds \right) \quad (j = 2, 3, \ldots; n = 1, 2, \ldots). \]

It is easy to show that \( H^{(n)}(0) = 0 \) for \( i > 0 \). Hence, \( h^{(n)}(0) = 0 \) for \( i > 0 \), so that the above equation simplifies to

\[ \tilde{W}_j^{(n)} = -H^{(n)} + \sum_{r=0}^{n} \frac{s^{j+r-2}}{(j+r-2)!} \left( \int_{0}^{s} \frac{d^{j+r-2}}{ds^{j+r-2}} h^{(j+r-2)}(s) \tilde{W}_{j+r}(s) ds \right) \quad (j = 2, 3, \ldots; n = 1, 2, \ldots). \]
(Chap. 5, Ex. 97 b)

b. Defining \( F(t) = P(W > t | W > 0) \) we assume the Maclaurin series expansion

\[
F(t) = \sum_{n=0}^\infty \frac{t^n}{n!} F^{(n)}(t) \quad (F^{(0)} = 1).
\] (5)

Using our present notation, (12.4) is written

\[
F(t) = 1 - H(t) + \sum_{n=1}^{\infty} \frac{t^n}{n!} F^{(n)}(t).
\]

Repeated differentiation w.r.t. \( t \) yields

\[
F^{(n)}(t) = \frac{\partial H^{(n)}(t)}{\partial t} - \frac{\partial H^{(n-1)}(t)}{\partial t} \frac{\partial H(t)}{\partial t} + \frac{\partial H^{(n-2)}(t)}{\partial t} \frac{\partial H(t)}{\partial t} - \cdots
\]

for \( n = 1, 2, \ldots \). Hence,

\[
F^{(n)}(t) = -H^{(n)}(t) + \sum_{j=1}^{n} \frac{\partial H^{(j)}(t)}{\partial t} \frac{\partial H^{(n-j)}(t)}{\partial t} \left( H^{(n-j)}(t) \right) \quad (j = 1, 2, \ldots) \quad (6)
\]

\[Q^{(1)}_j(x) = \frac{i}{t} \int \left( \sum_{j=1}^{n} \left( \Pi^*_x P_j(x) + \sum_{j=1}^{n} \Pi^*_x P_j(x) \right) + \frac{t}{\Pi^*_x P_j(x)} \right) dH(x) \quad \quad (j = 1, 2, \ldots) \quad (12.7)
\]

Hence,

\[
Q^{(1)}_j = \frac{i}{t} \int \left( \sum_{j=1}^{n} \left( \Pi^*_x P_j(x) + \sum_{j=1}^{n} \Pi^*_x P_j(x) \right) + \frac{t}{\Pi^*_x P_j(x)} \right) dH(x) \quad \quad (j = 1, 2, \ldots)
\]

The integrand (in parentheses) is the equilibrium probability of departure state \( j-1 \), given service time \( t \). Thus the integration results in the unconditional probability of departure state \( j-1 \), that is, \( \Pi^*_x P_j \). We conclude that

\[
Q^{(1)}_j = \frac{i}{t} \Pi^*_x P_j \quad \quad (j = 1, 2, \ldots) \quad (7)
\]

Repeated differentiation of (12.7) w.r.t. \( t \) yields

\[
Q^{(2)}_j = -\frac{1}{t^2} \sum_{m,n} \left( \Pi^*_x P_j(x) \right) \frac{\partial H^{(m)}(t)}{\partial t} \frac{\partial H^{(n)}(t)}{\partial t} \left( \Pi^*_x P_j(x) \right) \quad \quad (j = 1, 2, \ldots)
\]

whereby

\[
Q^{(2)}_j = -\frac{1}{t^2} \sum_{m,n} \left( \Pi^*_x P_j(x) \right) \frac{\partial H^{(m)}(t)}{\partial t} \frac{\partial H^{(n)}(t)}{\partial t} \left( \Pi^*_x P_j(x) \right) \quad \quad (j = 1, 2, \ldots) \quad (8)
\]
Chapter 5, Exercise 28

(Carter and Cooper [1972]) - cf. Ex 32 of Chap 3

The exercise is an application of the results of Exercise 2.7 on the M/M/1 random service queue. Thus we assume

\[ H(x) = 1 - e^{-xy}. \] (1)

Clearly, \( \hat{H}(x) = H(x) = 1 - e^{-xy} \), whereby

\[ \hat{H}^{(1)}(x) = \frac{x}{y} e^{-xy}, \] (\*)

\[ \hat{H}^{(2)}(x) = -\frac{1}{y^2} e^{-xy}. \] (\*\*)

a) By Eq. (6) of Exercise 2.7,

\[ F^{(1)} = -\frac{1}{y} + \frac{\phi}{y} e^{-\frac{1}{y} \phi} Q_{j}^{(0)}(\phi) \] \( W_j^{(0)} \)

By (\*) and the fact that \( W_j^{(0)} - W_j^{(0)} = 1 \) for \( j \geq 2 \), then,

\[ F^{(1)} = -\frac{1}{y} + \frac{x}{y} e^{-\frac{1}{y} x} Q_{j}^{(0)}. \] (2)

b) By Eq. (7) of Exercise 2.7, \( Q_j^{(0)} = \frac{1}{\prod_{i=1}^{j} (1 - \pi_i)} \). For an M/M/1 queue, \( \prod_{i=1}^{j} (1 - \pi_i) = (1 - \phi) \phi^j \), where \( \phi = \lambda \mu. \) Thus

\[ Q_j^{(0)} = \frac{1}{\prod_{i=1}^{j} (1 - \pi_i)} \quad (j = 1, 2, \ldots). \] (\* \* \*)

By (2) and (\* \* \*),

\[ F^{(0)} = -\frac{1}{y} + \frac{1}{y} \frac{1 - \phi}{\phi} e^{-\frac{1}{y} \frac{1 - \phi}{\phi} \phi^j} \]

\[ = -\frac{1}{y} + \frac{1}{y} \frac{1 - \phi}{\phi} e^{-\frac{1}{y} \frac{1 - \phi}{\phi} \phi^j} - \frac{1}{y} \frac{1 - \phi}{\phi} \phi^j \]

\[ = -\frac{1}{y} \frac{1 - \phi}{\phi} \phi^j \frac{1}{y} \frac{1 - \phi}{\phi} \phi^j \]

Hence,

\[ F^{(0)} = -\frac{1}{y} \frac{1 - \phi}{\phi} \ln \frac{1}{1 - \phi}. \] (3)
(Chap 5, Ex. 28 a)

\[ F^{(a)} = -\tilde{H}^{(a)} + \sum_{j=1}^{\infty} \frac{2}{z+1} \left( q_i^{(a)} \tilde{W}_i^{(a)} + q_i^{(a)} W_i^{(a)} \right). \]

By \((\bullet \bullet)\) and the fact that \( \tilde{W}_i^{(a)} = \tilde{W}_i^{(0)} = 1 \) for \( \frac{1}{2} \leq z \), then,

\[ F^{(a)} = \frac{1}{z+1} + \sum_{j=1}^{\infty} \frac{2}{z+1} \left( q_j^{(a)} \tilde{W}_j^{(a)} + q_j^{(a)} \right). \quad (4) \]

\[ Q_j^{(a)} = -\frac{1}{z+1} \tilde{H}_j^{(a)} \left( \Pi_1^{(a)} P_j^{(a)} + \frac{1}{z+1} \tilde{H}_j^{(a)} P_j^{(a)} \right) \quad (j=1,2,\ldots). \quad (5) \]

As \( \rho_j(z) = \left( \frac{1}{\lambda_j} \right)^{1/2} e^{-\lambda_j z} \), we have \( P_j^{(a)} = 1 \) and \( P_j^{(a)} = 0 \) for \( z \geq 1 \). Substitution of these values for \( P_j^{(a)} \) and \( P_j^{(a)} \) as well as \( H_j^{(a)} = \frac{1}{z+1} \) into (5), but only for \( j = 2,3,\ldots \), we obtain

\[ Q_j^{(a)} = -\frac{1}{z+1} \tilde{H}_j^{(a)} \quad (j=2,3,\ldots). \quad (6) \]

By Eq. (4) of Exercise 27,

\[ \tilde{W}_j^{(a)} = -\tilde{H}_j^{(a)} + \frac{z+1}{z} \tilde{H}_j^{(a)} \tilde{W}_j^{(a)} \quad (j=2,3,\ldots). \]

By definition, \( \tilde{G}_0^{(a)} = \tilde{G}_0^{(a)} = H_0^{(a)} P_0^{(a)} \). As \( H_0^{(a)} = \frac{1}{z+1} \) and \( P_0^{(a)} = 1 \), we have \( \tilde{G}_0^{(a)} = \frac{1}{z} \). Also, \( \tilde{W}_0^{(a)} = \tilde{W}_0^{(a)} = 0 \), and \( \tilde{W}_0^{(a)} = \tilde{W}_0^{(a)} = 1 \) for \( \frac{1}{2} \leq z \). We conclude that

\[ \tilde{W}_j^{(a)} = \left\{ \begin{array}{ll} \frac{1}{z} & (j=0), \\ \frac{1}{z} + \frac{z-1}{z} & (j=1), \\ \frac{1}{z} & (j=2,3,\ldots). \end{array} \right. \quad (7) \]

or

\[ \tilde{W}_j^{(a)} = -\frac{1}{z+1} \quad (j=2,3,\ldots). \]
Substitution into (4) of the expressions that have been derived for $Q_2^m$, $Q_1^m$, and $W_2^m$, and subsequent reduction, give:

$$F^m = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{j=0}^{m-1} \left[ \frac{1 - \frac{1}{\varphi} \sum_{i=1}^{j+1} \varphi^i}{\sum_{i=1}^{j+1} \varphi^i} + \frac{1}{\lambda} (1 - \varphi) \prod_{i=1}^{j+1} (1 - \varphi^i) \right]$$

$$= \frac{1}{\lambda} \left[ 1 - \frac{1}{\varphi} \sum_{i=1}^{m+1} \varphi^i - (1 - \varphi) \prod_{i=1}^{m+1} \varphi^i \right]$$

$$= \frac{1}{\lambda} \left[ 1 - \frac{1}{\varphi} \sum_{i=1}^{m+1} \varphi^i + (1 - \varphi) \sum_{i=1}^{m+1} \varphi^i \right]$$

$$= \frac{1}{\lambda} (1 - \varphi) \left[ 2 - \frac{1}{\varphi} \sum_{i=1}^{m+1} \varphi^i \right].$$

Thus, as expected,

$$F^m = \frac{1}{\lambda} (1 - \varphi) \left[ 2 - \frac{1}{\varphi} \ln \frac{1}{1 - \varphi} \right]. \quad (8)$$

Also Eq. (6) of Exercise 32 of Chapter 3 expresses the conditional waiting time distribution function in terms of a Maclaurin series expansion, but for an M/M/1 random service queue. For $s = 1$ the formula specializes to

$$P(W > t | W > 0) = 1 + t F^m + t^2 F^m + \ldots$$

where

$$F^m = -\frac{1}{\varphi} \ln \frac{1}{1 - \varphi},$$

$$F^m = \frac{1}{\lambda} (1 - \varphi) \left[ 2 - \frac{1}{\varphi} \ln \frac{1}{1 - \varphi} \right],$$

in complete agreement with Eqs. (5) and (8).
The additional-conditioning-variable method for the M/D/1 random-service-queue (Cutter and Cooper [1972]).

We assume a service time equal to the constant \( \tau \), i.e.

\[
H(x) = \begin{cases} 
0 & \text{when } x < \tau, \\
1 & \text{when } x \geq \tau.
\end{cases}
\]  
(1)

By (1), \( dH(x) = 1 \) and \( dH(x) = 0 \) for \( x \neq \tau \). Inserting this into Eq. (12.7) we find that for \( 3 > \tau \) is \( Q'_3(t) = 0 \) whereas for \( 3 \leq \tau \),

\[
Q'_3(t) = \frac{1}{2} \int_{x=0}^{\tau} (\Pi^*_x \beta_x(t)) + \frac{1}{2} \int_{x=0}^{\tau} \Pi^*_x \beta_x(t) dH(x)
- \frac{1}{2} \left( \Pi^*_x \beta_x(\tau) + \int_{x=0}^{\tau} \Pi^*_x \beta_x(t) dt \right).
\]

We conclude that

\[
Q'_3(t) = \begin{cases} 
\frac{1}{2} \Pi^*_x & \text{when } 3 \leq \tau, \\
0 & \text{when } 3 > \tau.
\end{cases}
\]  
(2)

It is obvious, and follows also from Eqs. (11.2) and (10), that remaining service time has distribution function

\[
\tilde{H}(t) = \begin{cases} 
\frac{1}{\tau} & \text{when } t < \tau, \\
1 & \text{when } t \geq \tau.
\end{cases}
\]

According to (12.9),

\[
P(W \gt t | W > 0) = 1 - \tilde{H}(t) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{0}^{t} \tilde{W}_k(t-1) Q'_k(1) dt.
\]

Substitution of the above expressions for \( \tilde{H}(t) \) and \( Q'_3(t) \) yields

\[
P(W \gt t | W > 0) = 1 - \frac{1}{\tau} + \frac{1}{\tau} \sum_{k=1}^{\infty} \int_{0}^{t} \tilde{W}_k(t-1) dt, \quad (0 \leq t < \tau), \tag{3}
\]

\[
P(W \gt t | W > 0) = \frac{1}{\tau} \sum_{k=1}^{\infty} \int_{0}^{t} \tilde{W}_k(t-1) dt, \quad (t \geq \tau). \tag{4}
\]
(Chap. 5, Ex. 29.2)

[Text content regarding a mathematical derivation and calculations, likely involving integrals and probability distributions, is shown here in a natural form.]
For constant service time $= \tau$, for all $x$,
\[ \{ \bar{X}(x) = k \} \Leftrightarrow \{ k \geq x \times (N+1) \tau \} \Leftrightarrow k \geq \left\lceil \frac{x}{\tau} \right\rceil. \]

Hence,
\[ P(\bar{X}(x) = k) = \begin{cases} 
1 & \text{when } k \geq \left\lceil \frac{x}{\tau} \right\rceil, \\
0 & \text{otherwise}. 
\end{cases} \quad (9) \]

According to (12.13),
\[ \tilde{W}_j(x) = \sum_{k=0}^{\infty} \tilde{W}_{j,k}(x) P(\bar{X}(x) = k) \quad (j = 2, 3, \ldots). \]

Using (9) we find
\[ \tilde{W}_j(x) = \tilde{W}_{j,\left\lceil \frac{x}{\tau} \right\rceil}(x) \quad (j = 2, 3, \ldots). \quad (10) \]

By setting $k = \left\lceil \frac{x}{\tau} \right\rceil$ and $k-1 = \left\lceil \frac{x-1}{\tau} \right\rceil$ in Eq. (8) it is obtained
\[ \tilde{W}_{j,\left\lceil \frac{x}{\tau} \right\rceil}(x) = \sum_{i=0}^{\infty} p_i(\tau) \tilde{W}_{j,i,\left\lceil \frac{x-i}{\tau} \right\rceil}(x-i) \quad (j \geq 2, \ldots). \]

The application of (10) leads to
\[ \tilde{W}_j(x) = \sum_{i=0}^{\infty} p_i(\tau) \tilde{W}_{j,\left\lceil \frac{x-i}{\tau} \right\rceil-i}(x-i) \quad (j \geq 2, \ldots, \tau x). \quad (11) \]

It is evident that at the start of service at $T$, any waiting customer will have to wait at least $\tau$ time units. That is,
\[ \tilde{W}_j(x) = 1 \quad (j \geq 2, \ldots, x > \tau). \quad (12) \]

Equations (11) and (12) give $\tilde{W}_j(x)$ for $j = 2, 3, \ldots$ and all $x$.

For $x = \tau$, we have $\tilde{W}(x) = 1$. Furthermore, $dH(\tau) = 1$ for $3 \tau$, and $dH(3) = 0$ for $3 \tau$. Substitution of these values into Eq. (12.10) also results in Eq. (11).
Chapter 5, Exercise 30

The M/G/1 queue with gating.

We shall not give all the details of the proof since it is precisely as the proof of (13.14) for the cyclic queue except that $R(x)$ replaces $B(x)$ and $\eta(s)$ replaces $\beta(s)$.

The explanation for this analogy is simple enough. In both cases we seek the mean $\bar{n}$ of the state distribution of an embedded Markov chain. Let $i$ be the state variable.

If $j \geq 1$, let $i^* = i$, and if $j = 0$ let $i^* = 1$. When $j \geq 1$, service begins (is resumed) right away, but when $j = 0$ service will not begin until a customer arrives. Denote by $t^*$ the time from start of service until next epoch of the embedded Markov chain. In the case of the cyclic queue, $t^*$ is a $i^*$-busy period, in the case of the queue with gating, $t^*$ is the sum of $i^*$ service times. In either case, the state at next epoch will be the number of customers arriving during the mentioned interval of length $t^*$ either at the other queue (cyclic queue) or at the same queue (queue with gating).

For the M/G/1 queue with gating,

\[ P(k) = \sum_{j=0}^{\infty} P(j) \frac{\lambda(1-\lambda x)}{\lambda} e^{-\lambda t} dH(x) + P(0) \frac{\lambda(1-\lambda x)}{\lambda} e^{-\lambda t} d\bar{H}(x) \quad (k=0,1,\ldots) \quad (13.1 a) \]

\[ g(x) = \sum_{k=0}^{\infty} P(k) x^k \]

\[ g(\eta(\lambda-\lambda x)) = g(x) = P(0)[1-\eta(\lambda-\lambda x)] \quad (\lambda \neq 1, \lambda x) \quad (13.5 a) \]

\[ 2 x \eta(x) - 2 x \eta(\lambda-\lambda x) - \eta(\lambda-\lambda x) = (1-z x) \quad (z \neq 1, \lambda x) \quad (13.6 a) \]

\[ P(0) = \frac{1}{1-\sum_{j=1}^{\infty} (1-x j)} \quad (13.11 a) \]

\[ -\lambda \eta(\lambda) \bar{n} - \bar{n} = \lambda P(0) \eta(\lambda) \quad (13.12 a) \]

By (13.11 a) and the definition $x_j = 2 x(j)i^* = i^* j^* \ldots$,

\[ P(n) = \frac{1}{1-\sum_{j=1}^{\infty} (1-x j)} \quad (\star) \]

The mean number $\bar{n} = n(0)$ of customers in the system when the gate opens is found from (13.13 a), (\star), $-\eta(\lambda) = \tau$ and $\lambda = \rho$:

\[ \bar{n} = \frac{\rho}{1-\rho} \frac{1}{1-\sum_{j=1}^{\infty} (1-x j)} \quad \blacksquare \]
Chapter 5, Exercise 31

Show that Equation (44) of Cooper [1969] is incorrect.

As hinted, the error in Eq. (44) is introduced in Eq. (42). It is true as stated immediately after Eq. (42) that \( P_i(n) = P_i(n)/[1 - P_i(0)] \) is the probability that \( (n-1, \ldots) \) customers wait in queue \( i \) when the gate closes, given \( n \geq 1 \). However, it is not true, as implied by Eq. (42), that \( P_i(n) \) also is the probability that an arbitrary customer in queue \( i \), who did not arrive when the system was completely empty, will be a member of a group of \( n \) customers at the time the gate closes. The latter probability is proportional to both \( n \) and \( P_i(0) \). Hence, in Eq. (42) one should replace \( P_i(n)/[1 - P_i(0)] \) by

\[
\frac{n P_i(n)}{\sum_{n=1}^{\infty} n P_i(n)} = \frac{n P_i(n)}{\sum_{n=1}^{\infty} n P_i(n)}.
\]

It follows that the same substitution should take place in Eqs. (43) and (44). Equation (44) changes into

\[
\omega_i(s) = (1-p) + \frac{\sum_{n=0}^{\infty} n P_i(n)}{\sum_{n=1}^{\infty} n P_i(n)} \left( -\lambda_i + \lambda_i \eta_i(s) \right) \left( -\lambda_i + \lambda_i \eta_i(s) \right)^{n-1},
\]

which may be rewritten as

\[
\omega_i(s) = (1-p) + \frac{\sum_{n=0}^{\infty} n P_i(n)}{\sum_{n=1}^{\infty} n P_i(n)} \left( -\lambda_i + \lambda_i \eta_i(s) \right) \left( -\lambda_i + \lambda_i \eta_i(s) \right)^{n-1}.\]

Now,

\[
\frac{\partial}{\partial s} g_{i,n}(s,1,...,1) \big|_{s=1} = \frac{\sum_{n=1}^{\infty} n P_i(n)}{\sum_{n=1}^{\infty} n P_i(n)} \left( -\lambda_i + \lambda_i \eta_i(s) \right) \left( -\lambda_i + \lambda_i \eta_i(s) \right)^{n-1}.
\]

We conclude that, for \( i = 0, 1, \ldots, N-1 \),

\[
\omega_i(s) = (1-p) + \frac{\sum_{n=0}^{\infty} n P_i(n)}{\sum_{n=1}^{\infty} n P_i(n)} \left( -\lambda_i + \lambda_i \eta_i(s) \right) \left( -\lambda_i + \lambda_i \eta_i(s) \right)^{n-1}.
\]

\[\square\]
Chapter 5, Exercise 32

'Verify that, for a Poisson input, Eq.(14.19) reduces to $P(W>0) = C(0,\omega)'$

With Poisson input at rate $\lambda$ the Laplace-Stieltjes transform of the interarrival time distribution function is

$$Y(a) = \frac{\lambda}{\lambda + a}.$$  

Inserting $\lambda/\mu$ for $\omega$ in the right-hand side of (14.12) we get

$$Y(1 - \lambda/\mu)\mu) = Y(\mu - \lambda) = \lambda/\mu,$$  

which proves that for the $M/M/\infty$ queue

$$\omega = \frac{\lambda}{\mu} = \frac{a}{a}.$$  

Also, (14.15) becomes

$$Y_i = Y(\mu) = \frac{\lambda}{\lambda + \mu} \quad (j = 0, 1, ..., s),$$  

and (14.16) becomes

$$C_i = \frac{1}{Y_i} - Y_i = \frac{a_i}{\mu} \quad (i = 1, 2, ..., s).$$  

By (14.14), (1), (2) and (3), Eq. (14.17) becomes

$$P(W>0) = \frac{\lambda}{\lambda + \omega} = \left[\left(1 - \frac{1}{\omega}\right)^{-1} + \frac{\lambda}{\mu} c_1(1 - \rho)\right]^{-1} \left[\mu^{-1} \left\{\frac{\lambda}{\mu} - \frac{1}{\omega}\right\}^{-1} + \left\{1 + \left(\frac{1}{\rho} - 1\right)\frac{\lambda}{\mu} \frac{\lambda}{\mu} - \frac{1}{\omega}\right\}^{-1} \left[\frac{1}{\omega} \frac{\mu}{\omega} \right]^{-1} \right].$$

Further rewriting gives

$$P(W>0) = \frac{\lambda^2}{\mu^2} \frac{\lambda}{\mu} \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2} \frac{\lambda}{\mu} \frac{\lambda}{\mu} \frac{1}{\omega}.$$  

The rightmost term is precisely the formula for $C(0,\omega)$ [Eq.(14.8) of Chap.3]. Thus,

$$P(W>0) = C(0,\omega).$$
'Prove that in a GI/M/s queue...'. Ex 29 of Chap 3

\[ P = \text{equilibrium probability that a blocked customer will still be waiting in the queue when the next customer arrives} \]

\[ \tau_i = \text{conditional probability that a blocked customer will still be waiting in the queue when the next customer arrives, given arrival state } s + i \ (i = 0, 1, \ldots). \]

Clearly, with service in order of arrival,

\[ \tau_i = \int_o^\infty \sum_{j=0}^i jw(x) e^{-sx} dG(x). \]

Hence,

\[ P = \sum_{i=0}^\infty \tau_i P(Q = \frac{i}{s} \mid X > 0) \]

\[ = \sum_{i=0}^\infty \left( \int_0^\infty \int_0^\infty \left( \sum_{j=0}^i jw(x) \right) e^{-sx} dG(x) \right) (1-\omega)^i \quad [\text{by (14.14)}] \]

\[ = \int_0^\infty e^{-sx} \int_0^\infty \sum_{j=0}^i \frac{(sx)^j}{j!} (1-\omega)^i dG(x) \]

\[ = \int_0^\infty e^{-sx} \sum_{j=0}^\infty \frac{(sx)^j}{j!} \int_0^\infty (1-\omega)^i dG(x) \]

\[ = \int_0^\infty e^{-sx} \sum_{j=0}^\infty \frac{(sx)^j}{j!} \omega^j dG(x) \]

\[ = \int_0^\infty e^{-(1-\omega)sx} dG(x) \]

\[ = \gamma((1-\omega)sx). \]

But, by (14.11), \( \omega = \gamma((1-\omega)sx). \) Thus,

\[ P = \omega. \]

This generalizes the result of Exercise 29 of Chapter 3. \( \square \)
Chapter 5, Exercise 34

'Verify Equations (14.7) and (14.8)'

Equation (14.7): \(1 \leq s \leq i, \ i+1 \geq 0\)

Given interarrival time \(x\), the binomial distribution

\[
p_i^Y(x) = \binom{s}{s-i} [\mu x]^s (1-\mu x)^{s-i} \quad (0 \leq x \leq i+1),
\]

since \(p_i^Y(x)\) is the probability of \(s-i\) successes (completions) in \(i+1\) trials, each with probability of success equal to \(\mu x\).

As \(p_i^Y = \int_0^\infty p_i^Y(x) \, dQ(x)\),

\[
p_i^Y = \int_0^{\infty} e^{-\mu x} \left(1-e^{-\mu x}\right)^{i+1} \, dG(x) \quad (i \geq s \geq 0).
\]

Equation (14.8): \(i \geq s \geq 0, \ i+1 \geq 0\)

Given interarrival time \(x\), the next arrival state will be \(j\) if and only if (i) at some time \(Y\), \(0 < Y < x\) (set \(T_i = 0\)), a service completion will leave exactly \(s\) customers in the system, and (ii) \(s-j\) service completions occur in the time interval \(Y\).

In a queue with departure rate \(\mu y\) (in effect as long as all servers are busy) the time \(Y\) until the \((i+1-s)\)th service completion, which will result in state \(s\), has an Erlangian distribution with the density function, by (55b) of Chapter 2,

\[
f(Y) = f(y) = (\mu y)^{i-s} e^{-\mu y} \mu^{i-s}.
\]

For a given \(Y = y < x\), the probability of \(s-j\) service completions during the remaining interarrival interval, of length \(x-y\), equals

\[
g_i^Y(x-y) = \binom{s}{s-j} (\mu x-y)^{s-j} (1-\mu x-y)^{j-s}.
\]

Hence, since \(p_i^Y(x) = \int_0^\infty g_i^Y(x-y) \, f(y) \, dy\) and \(p_i = \int_0^\infty p_i^Y(x) \, dG(x)\),

\[
p_i = \int_0^\infty \left(\frac{\mu x-y}{\mu x}\right)^{s-j} \left(\frac{1-\mu x-y}{1-\mu x}\right)^{j-s} e^{-\mu x} \mu y dy \, dG(x) \quad (i \geq s \geq 0, \ i+1 \geq 0).
\]
Chapter 5, Exercise 35

Derivation of (14.14) and the probabilities $\mathbb{P}_j, \mathbb{P}_{j-1}, \ldots, \mathbb{P}_{-1}$.

For the GI/M/s queue, it has been shown that $\mathbb{P}_j = A \mathbb{P}_{j-1}$ for $j \geq 2$, see (14.10), and now we must prove that $\mathbb{A}$ is given by (14.14). At the same time, we derive a formula for $\mathbb{P}_j$ for $j = 0, 1, \ldots, s-2$.

Let

$$U(z) = \sum_{i=0}^{\infty} \mathbb{P}_i z^i.$$  

Substitution of $\mathbb{P}_j = \sum_{i=0}^{\infty} \mathbb{P}_i \mathbb{P}_j$ by (14.3), and change of the order of summation give

$$U(z) = \sum_{i=0}^{\infty} \mathbb{P}_i \sum_{j=0}^{i} \mathbb{P}_j z^i,$$

or,

$$U(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \mathbb{P}_j \mathbb{P}_i z^i + \sum_{i=0}^{\infty} \sum_{j=i}^{i} \mathbb{P}_j \mathbb{P}_i z^i.$$  

Calculation of $G = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \mathbb{P}_j \mathbb{P}_i z^i$.

Since $p_j = 0$ for $j > i+1$,

$$G = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \mathbb{P}_j \mathbb{P}_i z^i = \sum_{i=0}^{\infty} \mathbb{P}_{i} \sum_{j=0}^{i} \mathbb{P}_j z^i.$$  

The $\mathbb{P}_i$'s for $i \leq s-1$ and $j \leq i+1$ are given by (14.7). Substitution of (14.7) and interchange of summations and integration yield

$$G = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \mathbb{P}_j \mathbb{P}_i z^i.$$  

The inner sum is the probability generating function of a binomial variable and equals $(p+qz)^i$, where $p = e^{-\mu s} q = 1 - e^{-\mu s}$. Thus,

$$G = \int_{0}^{\infty} (1-e^{-\mu x} + 2e^{-\mu x}) \mathbb{P}_j (1-e^{-\mu x} + 2e^{-\mu x}) dG(x).$$  

By definition of $U(z)$, then

$$G = \int_{0}^{\infty} (1-e^{-\mu x} + 2e^{-\mu x}) U(1-e^{-\mu x} + 2e^{-\mu x}) dG(x).$$
By (14.6) or (14.7), \( p_{n-1,n} = \int_0^{\infty} e^{-\omega x} dG(x) \), and by (14.10), \( \Pi_{m-1} = A \omega^{m-1} \). Hence,

\[
S_{2,n} = p_{n-1,n} \Pi_{n-1} z^n = A \omega^{n-1} z^n \int_0^{\infty} e^{-\omega x} dG(x).
\]

Thus, \( S = S_1 - S_2 \),

\[
S = \int_0^{\infty} (1 - e^{-\omega x}) \mu U(-e^{-\omega x} + 2e^{-\omega x}) dG(x) - A \omega^{n-1} z^n e^{-\omega x} dG(x) \quad (**)
\]

Calculation of \( T = \sum_{j=0}^{\infty} \sum_{k=0}^{j} p_{j,k} \Pi_j \Pi_k \)

The \( p_{j,k} \)'s for \( i \geq n \) and \( j \geq n \) are given by (14.8). Substitution of (14.8) and interchange of summations and integrations yield

\[
T = \int_0^{\infty} \left( \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x).
\]

Substitution of \( \Pi_j = A \omega^{j-1} \), by (14.10), rewriting and simplification give

\[
T = A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

The last integral on the right-hand side reduces to

\[
\int_0^{\infty} e^{-\omega x y} (e^{\omega x y} - 1) dG(x) = \omega \int_0^{\infty} \left[ e^{-(\omega x y) + \omega x y} - e^{-\omega x y} \right] dG(x)
\]

\[
= 1 - \omega \int_0^{\infty} e^{-\omega x y} dG(x), \quad \text{[by (14.11)]}
\]

so that

\[
T = A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]

\[
= A \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left( e^{\omega x y} \left( \frac{1}{(\omega y)^{j+k+1}} \int_0^{\infty} e^{-\omega x y} \frac{z}{dG(x)} dy \right) \right) dG(x)
\]
By \((*)\), \((***)\), \((***)\), and the definitions of \(S\) and \(T\),

\[
U(z) = \int_0^\infty \left[ \left( 1 - e^{-\mu x} - z e^{-\nu x} \right) \left( 1 - e^{-\mu x} - z e^{-\nu x} \right) \right] d\phi(x)
+ \int_0^\infty \left[ \left( e^{-\mu y} - e^{-\nu y} + z e^{-\nu y} \right) \right] d\phi(y)
- A \frac{e^{-\nu x}}{x}.
\]

By \((1)\) and \((11,10)\),

\[
U(1) = \frac{1}{2\pi} \sum_{j=1}^\infty \Re \{ j \} = 1 - \sum_{j=1}^\infty \frac{e^{-\mu j}}{j} = 1 - \sum_{j=1}^\infty \frac{e^{-\nu j}}{j} = 1 - \sum_{j=1}^\infty \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} = 1 - \sum_{j=1}^\infty \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j}.
\]

Hence,

\[
U(1) = 1 - \sum_{j=1}^\infty \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} = 1 - \sum_{j=1}^\infty \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} = 1 - \sum_{j=1}^\infty \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j}.
\]

For \(j = 0, 1, \ldots, n-1\) let

\[
U_j = \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} = \frac{1}{j!} \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} = \frac{1}{j!} \left( \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} \right).
\]

with \(U_0 = U(1)\), and define

\[
U_j = \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} = \frac{1}{j!} \left( \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} \right), \quad (j = 0, 1, \ldots, n-1).
\]

By definition, \(U_0 = \frac{1}{1+1} U_1 = U(1)\). By \((3)\), therefore,

\[
U_0 = 1 - \sum_{j=1}^\infty \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j}.
\]

Repeated differentiation of \((2)\) gives

\[
U^{(j)}(z) = \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} \left( \frac{e^{-\mu j}}{j} \frac{e^{-\nu j}}{j} \right), \quad (j = 1, 2, \ldots, n-1).
\]
(Chap. 5, Ex. 35 c)

Hence,

\[ U_j = \frac{Y_j(\mu)}{s_j} = U_j \gamma(\mu) + U_j \gamma(\mu) + \gamma(\mu) \]

\[ + A(\frac{5}{4}) \int_0^\infty e^{-\mu x} x^\frac{a}{2} \gamma(\omega - a - \omega y) dy dG(x) \]

\[ - A(\frac{5}{4}) \quad (j = 1, 2, ..., a-1). \]

Substituting

\[ \int_0^\infty x^\frac{a}{2} \gamma(\omega - a - \omega y) dy = \frac{a \gamma(\omega - a - \omega y)}{a \gamma(\omega - a - \omega y)} \]

and reducing, we obtain

\[ U_j = U_j \gamma(\mu) + U_j \gamma(\mu) - A(\frac{5}{4}) \frac{a}{(\omega - a - \omega y)} \int_0^\infty e^{-\mu x} x^\frac{a}{2} \gamma(\omega - a - \omega y) dy dG(x) \]

\[ - A(\frac{5}{4}) \quad (j = 1, 2, ..., a-1). \]

The two integrals are equal to, respectively, \( w \) (by \( \text{Eqn.} \)) and \( y(\mu) \).

Substitution of these values, simplification, and replacement of \( \gamma(\mu) \) by \( \gamma_j \) give

\[ U_j = U_j \gamma_j + U_j \gamma_j - A(\frac{5}{4}) \frac{a \gamma(\omega - a - \omega y)}{(\omega - a - \omega y)} \int_0^\infty e^{-\mu x} x^\frac{a}{2} \gamma(\omega - a - \omega y) dy dG(x) \]

\[ - A(\frac{5}{4}) \quad (j = 1, 2, ..., a-1), \]

from which it is obtained the difference equation

\[ U_j = U_{j-1} - \frac{A(\frac{5}{4})}{1-t_j} \frac{a \gamma(\omega - a - \omega y)}{(\omega - a - \omega y)} \int_0^\infty e^{-\mu x} x^\frac{a}{2} \gamma(\omega - a - \omega y) dy dG(x) \quad (j = 1, 2, ..., a-1). \]  

(6)

**d** Next we define \( C_j \) and

\[ C_j = \frac{1}{1-t_j} \quad (j = 1, 2, ..., a), \]

and divide by \( C_j \) on both sides of (6). The result is

\[ \frac{U_{j1}}{C_j} = \frac{U_{j1}}{C_j} - \frac{A(\frac{5}{4})}{C_j(1-t_j)} \frac{a \gamma(\omega - a - \omega y)}{(\omega - a - \omega y)} \int_0^\infty e^{-\mu x} x^\frac{a}{2} \gamma(\omega - a - \omega y) dy dG(x) \quad (j = 1, 2, ..., a-1). \]  

(7)
Assuming \( 0 \leq i \leq n-2 \), we add equations (7) for \( j = 1, \ldots, n-1 \), whereby we derive

\[
\frac{U_1}{C_1} = \frac{U_{n-1}}{C_{n-1}} + A \sum_{i=1}^{n-1} \frac{1}{C_i (1-\eta^i)} \left( \frac{\eta^{(i-1)n-C_i}}{\eta^{(i-1)n-C_i+1}} \right) (i = 0, 1, \ldots, n-2),
\]

\[
\frac{U_i}{C_i} = \left( \frac{U_{i+1}}{C_{i+1}} - \frac{A \omega^{-1}}{\omega^{1+n}} \right) + A \sum_{j=1}^{n-1} \frac{1}{C_{i-j} (1-\eta^j)} \left( \frac{\eta^{(i-j)n-C_i}}{\eta^{(i-j)n-C_i+1}} \right) (i = 0, 1, \ldots, n-2).
\]

By (1) and (4), \( U_{n-1} = \prod_{i=1}^{n-1} \), and by (14.10), \( \Pi_{n-1} = A \omega^{-1} \). By definition, \( C_n = C_{n-1} / (1-\eta) \). It follows that the term in parentheses vanishes. Furthermore, as it is easily verified, the above equation also holds for \( i = n-1 \). Our conclusion is that

\[
\frac{U_i}{C_i} = A \sum_{j=1}^{n-1} \frac{1}{C_{i-j} (1-\eta^j)} \left( \frac{\eta^{(i-j)n-C_i}}{\eta^{(i-j)n-C_i+1}} \right) (i = 0, 1, \ldots, n-1). \tag{8}
\]

Now set \( i = 0 \), make the substitutions \( U_0 = 1 - \frac{A}{\omega} \) and \( C_0 = 1 \), and solve Eq. (8) for \( A \):

\[
A = \left( \frac{1}{\omega} + \sum_{j=1}^{n-1} \frac{1}{C_{j-1} (1-\eta^j)} \left( \frac{\eta^{(j-1)n-C_j}}{\eta^{(j-1)n-C_j+1}} \right) \right)^{-1}. \tag{14.14}
\]

By definition, \( U(2) = (1) \) is a polynomial in \( z \) of degree \( n-1 \). Hence, \( U(z) \) will be represented exactly by a Taylor series expansion of degree \( n-1 \) at an arbitrary point \( z \). For \( x \neq 1 \) the representation is

\[
U(2) = \sum_{i=1}^{n-1} \frac{U_{i-1}}{i!} (z-1)^i. \tag{9}
\]

By (1), \( U(1) = \sum_{i=1}^{n-1} \frac{U_{i-1}}{i!} (z-1)^i \). Hence, \( U(1) = \prod_{j=1}^{n-1} \), or,

\[
\Pi_j = \frac{U(1)}{z^j} = \frac{1}{j!} \left( \frac{1}{(z-1)^j} \right) (j = 0, 1, \ldots, n-1). \tag{10}
\]

Differentiation of (9) \( \frac{dU(2)}{dz} = \sum_{i=1}^{n-1} \frac{U_{i-1}}{i} (z-1)^{i-1} \) \( j \) times leads to

\[
\frac{d^j U(2)}{dz^j} = \sum_{i=1}^{n-1} \frac{U_{i-1}}{i} (z-1)^{i-1} \frac{j!}{j!} (j = 0, 1, \ldots, n-1),
\]

whose substitution into (10) yields

\[
\Pi_j = \frac{d^j U(1)}{dz^j} (-1)^{j-1} z_j \quad (j = 0, 1, \ldots, n-1). \tag{11}\]
Chapter 5, Exercise 36

Show that for Poisson input this algorithm yields Equation (2) of Exercise 32 of Chapter 3.

In the case of the M/M/s queue with service in random order, $\omega = q (-\lambda /\mu w)$, so that (15.9) becomes

$$P(W > t | W > 0) = 1 + (1-p) \sum_{n=1}^{\infty} \frac{\mu^n}{\mu^n} W_0^n,$$

where, according to (15.14),

$$W_0^n = a_1^{(n)} + \sum_{i=1}^{n} \sum_{j=0}^{n} b_i^{(n)} W_0^{(n-i)-j},$$

Eq. (2) may be solved recursively for $n=1, 2, \ldots$, utilizing $W_0^0 = W_0^0 = 1$ and the definitions of $a_i(t)$ by (15.12), and $b_i(t)$ by (15.13). In the present case, where $G(t) = 1 - \epsilon e^{-\lambda t}$,

$$a_i(t) = \epsilon^{-i} (\omega + \mu \lambda) \left[ \frac{\mu^i}{\mu^i} (\omega \epsilon)^{i-k} \frac{1}{k!} \right],$$

$$b_i(t) = \mu \epsilon^{i} (\omega \epsilon)^{i-k} \frac{1}{k!},$$

from which $a_i^{(n)}$ and $b_i^{(n)}$ may be derived.

We shall determine all terms of the expansion (1) to the order of $\epsilon^t$. Thus we must find $W_0^{(n)}$ and $W_1^{(n)}$ for $n=0, 1, \ldots$. By (2), this requires calculation of $a_1^{(n)}$ for $n=1, 2, \ldots$, and of $b_1^{(n)}$ for $n=0, 1, \ldots$.

Calculation of $a_1^{(n)}$ and $b_1^{(n)}$

Differentiating (5) once, we obtain

$$a_0^{(2)}(t) = -\lambda \epsilon \omega \mu \epsilon^{-1} (\omega \epsilon)^{k},$$

$$a_0^{(3)}(t) = -\lambda \epsilon \omega \mu \epsilon^{-1} (\omega \epsilon)^{k} \left[ 1 - \frac{1}{1+\epsilon} (\omega \epsilon)^{k-1} \right],$$

$$a_0^{(4)}(t) = -\lambda \epsilon \omega \mu \epsilon^{-1} (\omega \epsilon)^{k} \left[ 1 - \frac{1}{1+\epsilon} (\omega \epsilon)^{k-1} \right] \frac{\lambda}{\lambda + \epsilon} \left[ \frac{\mu^i}{\mu^i} (\omega \epsilon)^{i-k} \right],$$

Setting $t=0$, we find $a_0^{(n)} = -\lambda \epsilon \omega \mu (1 - \frac{1}{1+\epsilon})$ for $n=0, 1, 2, \ldots$. Thus,

$$a_0^{(i)} = -\lambda \epsilon \omega \mu \left( \frac{q}{q+\frac{\lambda}{\mu}} \right),$$

(5)
Another differentiation results in
\[ a_{\lambda}^{(i)}(t) = (\lambda + \sigma \mu)^i e^{-\lambda t}, \]
\[ a_{\sigma}^{(i)}(t) = (\lambda + \sigma \mu)^i t e^{-\lambda t}, \]
\[ a_{\mu}^{(i)}(t) = (\lambda + \sigma \mu)^i t e^{-\lambda t}, \]
where \( R_{i,j}(t) \) is a polynomial of degree \( j-i \). It follows that
\[ a_{\lambda}^{(0)} = (\lambda + \sigma \mu), \]
\[ a_{\sigma}^{(0)} = (\lambda + \sigma \mu)^2 \left( 1 - \frac{1}{i+q} \right), \]
\[ a_{\mu}^{(0)} = (\lambda + \sigma \mu)^2 \left( 1 - \frac{1}{i+q} \right) + \frac{1}{(i+q)^2} \frac{A}{q+1}, \]
\( (q = 2, 3, \ldots) \).

A rewriting yields
\[ a_{\lambda}^{(i)} = \begin{cases} 
(\sigma \mu)^i (q^2 + 2q + 1) & (j = 0), \\
(\sigma \mu)^i (q^2 + 2q + 1) & (j = 1, 2, \ldots).
\end{cases} \]  
(6)

Calculation of \( b_{j+1}^{(n)} \) and \( b_{j+1}^{(n)} \).

By (4), clearly \( b_{j+1}^{(n)} = \lambda \) and \( b_{j+1}^{(n)} = 0 \) for \( n \geq 2 \). Thus,
\[ b_{j+1}^{(n)} = \begin{cases} 
\lambda & (j+1 = 0), \\
0 & (j+1 = 1, 2, \ldots).
\end{cases} \]  
(7)

Differentiation of (4) results in
\[ b_{j}^{(0)}(t) = -\lambda (\lambda + \sigma \mu) e^{-\lambda t}, \]
\[ b_{j}^{(n)}(t) = \lambda e^{-\lambda t} \left( (\sigma \mu)^{n-1} - (\sigma \mu)^n \right) (\lambda + \sigma \mu)^{n-1} \]
\( (n = 1, 2, \ldots) \).

It follows that \( b_{j}^{(1)} = -\lambda (\lambda + \sigma \mu), \)
\[ b_{j}^{(n)} = \lambda \sigma \mu, \]
\[ b_{j}^{(n)} = 0 \] for \( n \geq 2 \). Thus,
\[ b_{j}^{(n)} = \begin{cases} 
-\lambda (\lambda + \sigma \mu) & (j+1 = 0), \\
\lambda \sigma \mu & (j+1 = 1), \\
0 & (j+1 = 2, 3, \ldots).
\end{cases} \]  
(8)
(Chap. 5, Ex. 8.6 (cont’d))

Calculation of \( W_j^{(0)} \)

By (2),

\[
W_j^{(0)} = a_j^{(0)} + \frac{2\pi}{\kappa j} b_{j+1}^{(0)} W_{j+1}^{(0)} (j = 0, 1, \ldots).
\]

Substitution of \( W_j^{(0)} = 1 \), (5), and (7), leads to \( W_j^{(0)} = 2\mu (q + \frac{1}{2})^j + \lambda \) for \( j = 0, 1, \ldots \). Hence,

\[
W_j^{(0)} = -\frac{\mu}{\lambda} (q + 1)^j (j = 0, 1, \ldots) \quad (9)
\]

Calculation of \( W_j^{(3)} \)

By (2),

\[
W_j^{(3)} = a_j^{(3)} + \frac{2\pi}{\kappa j} b_{j+1}^{(3)} W_{j+1}^{(3)} + \frac{j\pi}{\kappa j} b_{j+1}^{(0)} W_{j+1}^{(0)} (j = 0, 1, \ldots),
\]

which by substitution of \( W_j^{(0)} = 1 \), (7) and (9) reduces to

\[
W_j^{(3)} = a_j^{(3)} - \frac{\pi}{\lambda} (q + \frac{1}{2}) + \frac{j\pi}{\lambda} (q + 1)^j (j = 0, 1, \ldots).
\]

Finally, application of (6) and (8) results in

\[
W_0^{(3)} = (\omega^2 + 2\pi + 1) - \frac{2\pi}{\lambda} - \lambda (q + \mu),
\]

\[
W_1^{(3)} = (\omega^2 + \frac{2\pi}{\lambda}) - \frac{\pi}{\lambda} \omega - \lambda (q + \mu) - \frac{1}{\lambda} \omega (q + \mu) (j = 1, 2, \ldots),
\]

or,

\[
W_j^{(3)} = \left( \frac{(q + \omega)^{j+1} (q + \omega)}{(q + \omega)^{j+1} (q + \omega)} \right) (j = 0),
\]

\[
W_j^{(3)} = \left( \frac{(q + \omega)^{j+1} (q + \omega)}{(q + \omega)^{j+1} (q + \omega)} \right) (j = 1, 2, \ldots). \quad (10)
\]

Calculation of \( P(\omega^2 + 1) \)

Equations (9) and (10) are precisely those derived previously in Exercise 32 of Chapter 3. As in that exercise, the substitution of (9) and (10) into (1) and subsequent reduction therefore yield Equation (6) of Exercise 32 of Chapter 3.
Chapter 6, Exercise 1

'Show that if \( U \) is uniform on \((0,1)\), then so is \( 1-U \).'

Assume that
\[
P\{U \leq u\} = u \quad (0 \leq u \leq 1).
\]
Hence,
\[
P\{U < u\} = u \quad (0 \leq u \leq 1).
\]
Now, \( \{U < u\} \Leftrightarrow \{1-U > 1-u\} \), so that
\[
P\{1-U > 1-u\} = u \quad (0 \leq u \leq 1),
\]
whereby
\[
P\{1-U \leq 1-u\} = 1-u \quad (0 \leq u \leq 1).
\]
Substituting \( u' = 1-u \) we obtain
\[
P\{1-U \leq u'\} = u' \quad (0 \leq u' \leq 1).
\]
Thus, \( 1-U \) is uniformly distributed on \((0,1)\) if \( U \) is.

Chapter 6, Exercise 2

'Let \( X \) have the Erlangian distribution function of order \( n \),

\[
F_X(x) = 1 - \sum_{k=0}^{n-1} \frac{x^k}{k!} e^{-\lambda x}.
\]

\( X \) may be interpreted as the sum of \( n \) independent exponential variables with parameter \( \lambda \), that is,

\[
X = X_1 + X_2 + \ldots + X_n,
\]

where

\[
F_{X_i}(x) = 1 - e^{-\lambda x} \quad (i=1,\ldots,n).
\]

But, by (2.7), \( X_i = -\frac{1}{n} \ln U_i \), where \( U_i \) is uniform on \((0,1)\). Hence,

\[
X = \sum_{i=1}^{n} \left( -\frac{1}{n} \ln U_i \right) = -\frac{1}{n} \ln (U_1U_2\cdots U_n).
\]
Chapter 6, Exercise 3

Derive Equation (4.18).

The variable under consideration is

\[ \hat{P}(n) = \frac{\bar{x}(n)}{\bar{y}(n) + \bar{x}(n)}, \]  

with \( \bar{x}(n) = \sum_{i=1}^{n} x_i / n \), \( \bar{y}(n) = \sum_{i=1}^{n} y_i / n \), where all \( x_i \)'s and \( y_i \)'s are independent variables. The service time \( X_i \) has a general distribution, \( E(X) = \tau \), \( \nu(X) = \sigma^2 \). The idle time \( Y_i \) has an exponential distribution, \( E(Y) = \lambda^2 \), \( \nu(Y) = \lambda^2 \).

Now define the distribution function

\[ F_{\hat{P}}(t) = P(\hat{P}(n) \leq t). \]  

Substitution of (1) into (2) gives

\[ F_{\hat{P}}(t) = P\left( \frac{\bar{x}(n)}{\bar{y}(n) + \bar{x}(n)} \leq t \right). \]

Defining \( Z_i(t) = \frac{x_i - \frac{1}{n} \bar{x}(n)}{\frac{1}{n} \bar{y}(n)} \) and \( Z(n,t) = \sum_{i=1}^{n} Z_i(t) / n = \frac{\bar{x}(n) - \frac{1}{n} \bar{x}(n)}{\frac{1}{n} \bar{y}(n)} \), (3) may be written

\[ F_{\hat{P}}(t) = P\{ Z(n,t) \leq 0 \}. \]

We find easily that mean and variance of \( Z(n,t) \) are

\[ E(Z(n,t)) = \tau - \frac{1}{n} \bar{x}(n) \],  
\[ \nu(Z(n,t)) = \frac{\sigma^2}{n} \left[ \frac{(\lambda^2 - \tau^2)}{\lambda^2} \right]. \]

By the central limit theorem \( Z(n,t) \) is asymptotically normal distributed:

\[ \lim_{n \to \infty} P(\frac{Z(n,t)}{\sqrt{\nu(Z(n,t))}} \leq x) = \Phi \left( \frac{x - E(Z(n,t))}{\sqrt{\nu(Z(n,t))}} \right). \]

Setting \( x = 0 \), and substituting (5) and (6) we derive

\[ \lim_{n \to \infty} P(\frac{Z(n,t)}{\sqrt{\nu(Z(n,t))}} \leq 0) = \Phi \left( \frac{\tau - \frac{1}{n} \bar{x}(n)}{\frac{\sigma^2}{n} \left( \frac{(\lambda^2 - \tau^2)}{\lambda^2} \right)^{1/2}} \right). \]

By (4),

\[ \lim_{n \to \infty} F_{\hat{P}}(t) = \Phi \left( \frac{\tau - \frac{1}{n} \bar{x}(n)}{\frac{\sigma^2}{n} \left( \frac{(\lambda^2 - \tau^2)}{\lambda^2} \right)^{1/2}} \right). \]
Chapter 6, Exercise 4

Consider a simulation of the single-server Erlang loss model with constant service times.

Our two estimates \( \hat{\Pi}(n) \) and \( \hat{P}(n) \) of the loss probability \( \Pi \), based on a simulation of \( n \) cycles, have been shown to be asymptotically normal:

\[
F_{\hat{\Pi}(n)}(t) = \Phi\left(\frac{1 - \frac{t}{\tau}}{\sqrt{n} (\frac{\tau}{\tau^2 + a^2})}\right),
\]

\[
F_{\hat{P}(n)}(t) = \Phi\left(\frac{1 - \frac{t}{\tau}}{\sqrt{n} (\frac{\tau}{\tau^2 + a^2})}\right)
\]

(a) First, observe that \( \Pi = \frac{\tau}{\tau^2 + a^2} = 1 - a^2 \), whereby \( a = \frac{\tau}{\tau^2 + a^2} \).

Also, as \( n \rightarrow \infty \), both \( \hat{\Pi}(n) \) and \( \hat{P}(n) \) converge in probability to \( \Pi \). It is therefore obvious, and may be proved rigorously, that for any \( \varepsilon > 0 \) there exists an \( n_0 \) such that

\[
\Phi\left(\frac{1 - \frac{t}{\tau}}{\sqrt{n} (\frac{\tau}{\tau^2 + a^2})}\right) - \Phi\left(\frac{1 - \frac{t}{\tau}}{\sqrt{n} (\frac{\tau}{\tau^2 + a^2})}\right) < \varepsilon
\]

for all \( t, 0 < t < 1 \), and \( n > n_0 \). This means that asymptotically \( \hat{P}(n) \)'s distribution function may also be expressed

\[
F_{\hat{P}(n)}(t) = \Phi\left(\frac{1 - \frac{t}{\tau}}{\sqrt{n} (\frac{\tau}{\tau^2 + a^2})}\right).
\]

(b) Now assume constant service times, that is \( \sigma^2 = 0 \), and \( n = 100 \).

Substituting these values and \( a = \frac{\tau}{\tau^2 + \sigma^2} \) into (4.11) and (4.18) we find

\[
F_{\hat{\Pi}(100)}(t) = \Phi\left(10(1 - \frac{t}{\tau})\frac{\tau}{\tau^2 + \sigma^2}\right),
\]

\[
F_{\hat{P}(100)}(t) = \Phi\left(10(1 - \frac{t}{\tau})\frac{\tau}{\tau^2 + \sigma^2}\right)
\]
(Chap 6, Ex. 4)

Hence we obtain the approximation formulas
\[ P(\pi - 0.05 < \hat{\pi}(100) < \pi + 0.05) = \Phi(\hat{u}_1) - \Phi(\hat{u}_2), \]
\[ P(\pi - 0.05 < \hat{\pi}(100) < \pi + 0.05) = \Phi(\hat{u}_3) - \Phi(\hat{u}_4), \]
where
\[ u_1 = \frac{10}{1 - \pi} \left[ \frac{\pi + 0.05}{\pi - 0.05} - 1 \right] \sqrt{\frac{\pi - \pi^2}{\pi}}, \]
\[ u_2 = -10 \left[ \frac{\pi}{1 - \pi} - \frac{\pi - 0.05}{\pi - 0.05} \right] \sqrt{\frac{\pi - \pi^2}{\pi}}, \]
\[ u_3 = 10 \left[ 1 - \frac{\pi}{1 - \pi} \right] \frac{1 - \pi + 0.05}{\pi + 0.05}, \]
\[ u_4 = -10 \left[ \frac{\pi - 0.05}{1 - \pi} - 1 \right]. \]

Calculations

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(Chap. 6, Ex. 4 (cont'd))

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Diagram showing the relationship between $P_1$ and $T_1$.