Maximum-Shortest-Path (MSP) is Not Optimal for a General $N \times N$ Torus

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***Abstract*—A shortest-path routing is optimal if it maximizes the probability of reaching the destination from a given source, assuming that each link in the system has a given failure probability. An approximation for the shortest-path routing policy, maximum-shortest-path (MSP) routing was proposed by Wu [3]. Reference [3]**

- **shows that MSP is optimal in the mesh and hypercube networks,**
- **shows that MSP is at least suboptimal in the torus network,**
- **shows that MSP is optimal for 6×6 and 8×8 tori,**
- **conjectured that MSP is optimal for 2-D tori in general.**

This short paper shows that, contrary to the claims in [3], MSP is not optimal for a general $N \times N$ torus—specifically, MSP is not optimal for a 12×12 torus, and its optimal routing depends on the success probability.

***Index Terms*—Mesh, optimality, probability, shortest path routing, torus.**

ACRONYMS[1]

A	SPR policy
MSP	maximum shortest-path (policy)
SPR	shortest-path routing
Z^2	zig-zag
2-D	2-dimensional.

NOTATION

$N \times N$ torus (mesh)	2-dimensional torus (mesh) with N rows and N columns
$\{v_1, v_2, \ldots, v_m\}$	*eligible neighbor vector* of node v with respect to destination node u; m is the number of eligible neighbors of v
p	$\Pr\{$a message is successfully forwarded to a neighbor along a given link$\}$, also called: success probability
$S_A(v, u)$	maximum probability of delivery of a message from node v to node u under A
$P(v, u)$	number of shortest paths from node v to node u
$N_{\mathrm{sp}}(i, j)$	set of eligible neighbors for node (i, j).

Manuscript received November 18, 1999; revised July 27, 2000, October 26, and May 6, 2001. The work of J. Wu was supported in part by U.S. NSF Grant CCR 9900646 and Grant ANI 0073736. Responsible Editor: T. Kohda.

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Digital Object Identifier 10.1109/TR.2002.807240

[1]The singular and plural of an acronym are always spelled the same.

I. INTRODUCTION

2-D MESHES and 2-D tori (see Fig. 1) are commonly used mesh-connected networks to build multicomputer systems. In general the performance of such a multicomputer system depends on the end-to-end cost of communication mechanisms. Routing time of messages in terms of routing path length is one of the key critical factors in the performance of multicomputers.

If a message cannot continue along a shortest path (one with minimum routing path length) to its destination due to some link failures, it is discarded. For a message at any node v, SPR requires the message to be sent to an *eligible neighbor* of v with respect to destination u: a neighbor of v that is closer to the destination than v. A SPR policy specifies a preference ordering (among the $m!$ orderings) on the set of eligible neighbors for each node v, and a message arrives at v will always first attempt to go to the eligible neighbor with the highest preference in the chosen order.

Assumptions

1) Each link has a uniform failure probability of $1 - p$ $(0 \leq p \leq 1)$.

2) Due to the presence of faulty links, each SPR can only guarantee to deliver a message from a given source to its destination with a particular probability.

3) The higher the probability, the better the SPR policy.

4) Source $v = (i, j)$ (with $i, j \geq 0$) and destination $u = (0, 0)$ in a 2-D torus, and the maximum probability of delivering a message from (i, j) to (0, 0) is represented as $S((i, j), (0, 0))$, or simply $S(i, j)$ without causing confusion.

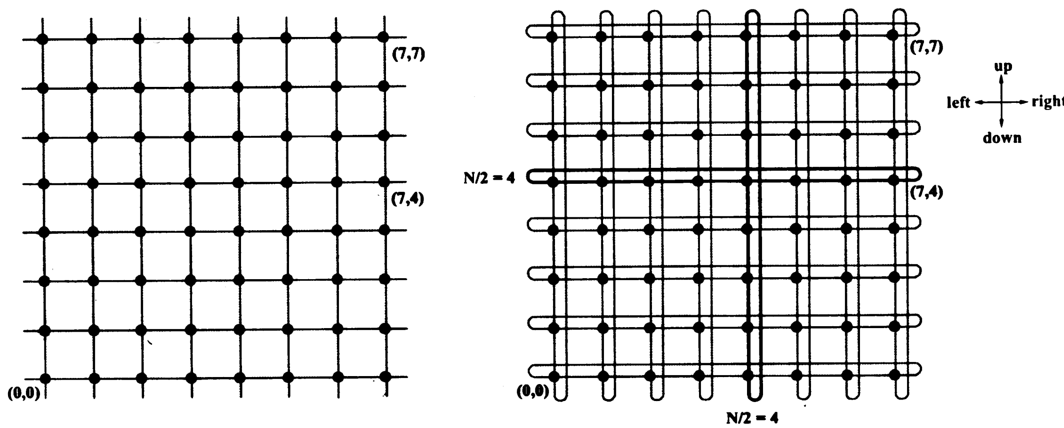
For a given A, one can explicitly calculate the $S_A(v, u)$. Because the given routing policy specifies a preference ordering on the set of eligible neighbors of every node, let $(v'_1, v'_2, \ldots, v'_m)$ be such a preference ordering on the eligible neighbors of the node v. The message first attempts to go to v'_1, and only if this fails, does it go to the next preferable eligible neighbor v'_2, *etc*. Therefore, the probability that this message is received by v'_i is $(1-p)^{i-1} \cdot p$. Then $S_A(v, u)$ can be recursively computed as follows:

$$S_A(v, u) = \sum_{i=1}^{m} (1-p)^{i-1} \cdot p \cdot S_A(v'_i, u)$$
$$S_A(v, v) = 1. \quad (1)$$

A SPR policy A^* is *optimal* if it maximizes the probability of delivering a message to destination u for a given source v:

$$S_{A^*}(v, u) = \max_A [S_A(v, u)].$$

S_{A^*} also satisfies the recursive relation (1).



(a) (b)

Fig. 1. Examples: (a) an 8 × 8 mesh, and (b) an 8 × 8 torus.

The SPR delivers messages through shortest paths between the source and destination nodes; in a SPR, only shortest paths are acceptable. Several SPR algorithms have been proposed for 2-D meshes and 2-D torus, including the Z^2 routing policy [1], and the MSP routing policy [3]. The Z^2 policy is one in which the routing message always moves toward the closest diagonal node (x, y) (with $x = y$). In the MSP policy, the routing message is always forwarded to an eligible neighbor from which there exists a maximum number of shortest paths to the destination

$$P(v, u) = \sum_{i=1}^{m} P(v_i, u)$$
$$P(v, v) = 1. \tag{2}$$

It has been shown that in a 2-D mesh or a binary hypercube, both Z^2 and MSP routing policies are optimal [1], [3]; and these two routing policies are the same for 2-D meshes and binary hypercubes. For a 2-D torus, however, it has been shown that the Z^2 policy is not optimal [2], [3]. In order to investigate whether or not the MSP routing is optimal for 2-D torus, [3] explicitly expresses the MSP routing in an $N \times N$ torus (N is a positive even number; see Section II of this paper). In addition, [3] shows that the MSP routing is at least suboptimal in a torus network, demonstrates that it is optimal for 6 × 6 and 8 × 8 tori, and conjectures that the MSP policy is optimal for 2-D tori in general. This paper shows that, contrary to [3], the MSP policy is not an optimal SPR policy for a 12 × 12 torus. The optimal routing depends on p.

II. MSP Routing is Not Optimal for a 12 × 12 Torus

An $N \times N$ torus can be defined as an $N \times N$ matrix of points with N rows and N columns. The set of points (nodes) can be represented as

$$\{(i, j)\colon 0 \le i, j \le N-1\}.$$

Each node (i, j) is connected through links to 4 neighbors:

$$(i-1, j), (i, j-1), (i+1, j), (i, j+1),$$

where addition and subtraction are modulo N. A link is wrap-around if it connects 2 nodes whose addresses differ by $N-1$ in a dimension. Fig. 1(b) is an example of an 8 × 8 torus together with 4 directions: right, up, left, down. A 2-D mesh is a 2-D torus without wrap-around links.

By symmetry, it can be assumed that source node (i, j) is such that $i, j \le N/2$ in an $N \times N$ torus. When N is odd, or when N is even and both $i \ne N/2$ and $j \ne N/2$, then node (i, j) has exactly the same eligible neighbors as its corresponding 2-D mesh; hence, any optimal SPR policy (Z^2 or MSP) for 2-D meshes is also optimal for 2-D tori for delivering a message from (i, j) to (0, 0). However, when N is even and, $i = N/2$ or $j = N/2$, unlike the case of 2-D meshes, the set of eligible neighbors for (i, j), $N_{\rm sp}(i, j)$, can have more than 2 elements

$$N_{\rm sp}(i, j) = \begin{cases} \{(i-1, j), (i+1, j)\} \\ \quad \text{if } i = \dfrac{N}{2} \quad \& \quad j = 0, \\ \{(i-1, j), (i+1, j), (i, j-1)\} \\ \quad \text{if } i = \dfrac{N}{2} \quad \& \quad j \ne \dfrac{N}{2} \ne 0, \\ \{(i, j-1), (i, j+1), (i-1, j)\} \\ \quad \text{if } i \ne \dfrac{N}{2} \ne 0 \quad \& \quad j = \dfrac{N}{2}, \\ \{(i-1, j), (i+1, j), (i, j-1), (i, j+1)\} \\ \quad \text{if } i = \dfrac{N}{2} \quad \& \quad j = \dfrac{N}{2}. \end{cases} \tag{3}$$

Therefore, the SPR policy that is optimal for $N \times N$ meshes might not be optimal in $N \times N$ tori with source node (i, j) being on the row $j = N/2$ or on the column $i = N/2$.

Consider an $N \times N$ torus, with N being a positive even number. The MSP routing [3] is an approximation of the optimal policy. At each step, an eligible neighbor with a maximum number of shortest paths to the destination, i.e., with the maximum P value, is selected. The following algorithm ensures that at each step, an eligible neighbor with the maximum P value is selected.

`Algorithm: MSP Routing`

When the source (i, j) is at the $i = N/2$ column, then there is a turning point at $(N/2, \lfloor N/4 \rfloor)$. There are 2 cases:

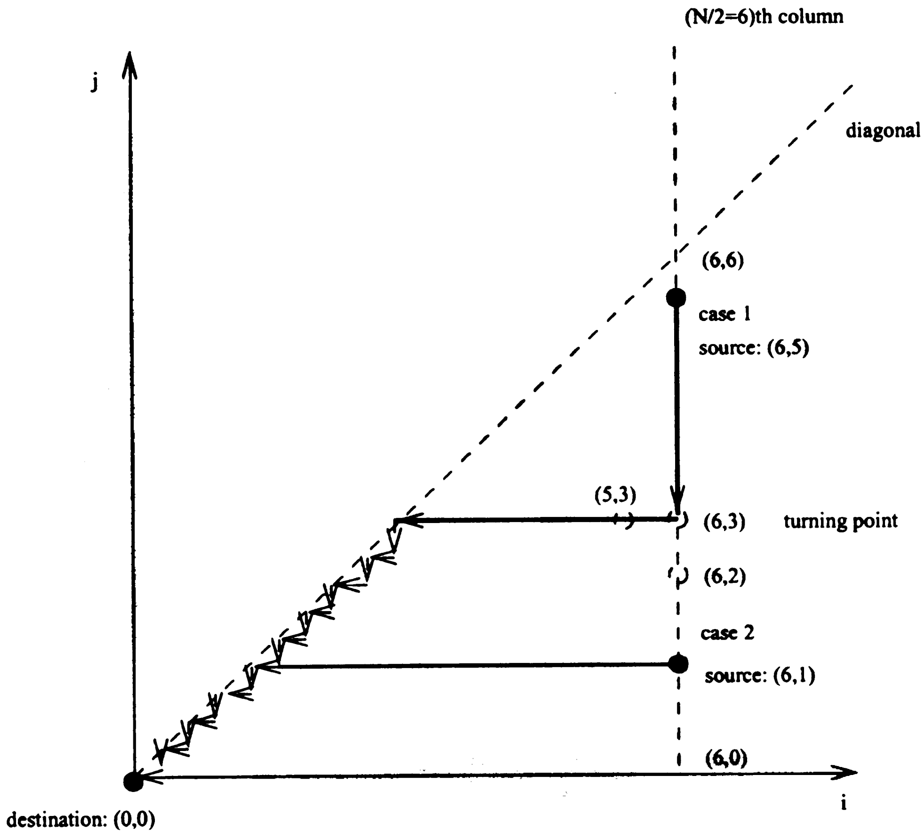


Fig. 2. Two MSP routing examples in a 12 × 12 torus.

1) $j > \lfloor N/4 \rfloor$ (the source node is above the turning point), then $P(i, j-1) > P(i-1, j)$. Thus, the routing message is forwarded **down** to $(i, j-1)$ as its first attempt. This process lasts until the message reaches $(i, \lfloor N/4 \rfloor)$, then follows the Z^2 routing: the message is forwarded **left** until it reaches the diagonal line L: $x = y$, and finally zig-zags around the diagonal line to reach the destination.

2) If $j \le \lfloor N/4 \rfloor$ (the source node is on or below the turning point) then $P(i, j-1) \le P(i-1, j)$. Thus, the routing message is delivered by following the Z^2 routing directly.

`End_Algorithm`

In this algorithm, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Fig. 2 shows 2 MSP routing examples in 12 × 12, where the source is on column $N/2 = 6$, and it has 3 eligible neighbors: case 1 represents the case when the source is above the turning point; case 2 represents the case when the source is (or below) the turning point. When the source has 4 eligible neighbors, the step 1 can be along either row or column, then the remaining steps are the same when the source is on row or column $N/2 = 6$.

It is not known if the MSP policy is optimal, i.e., if $P(v, u) \ge P(v', u)$ is equivalent to $S(v, u) \ge S(v', u)$, where v, v' are two eligible neighbors in a routing step. Reference [3]

- shows that the MSP policy is at least suboptimal in the torus network,
- demonstrates that it is optimal for 6 × 6 and 8 × 8 tori,
- conjectures that the MSP policy is optimal for 2-D tori in general.

This section now shows that, contrary to the claim in [3], the MSP policy is not optimal for a general $N \times N$ torus. Specifically, it shows that the MSP policy is not optimal for a 12 × 12 torus when the source is at the turn point (6, 3) and its optimal routing depends on the success probability p. Two facts are now established.

Fact 1: For a 12 × 12 torus,

$$S(5, 3) - S(6, 2) = p^9 \cdot (1-p)^2 \cdot (7 - 9p + p^2).$$

By direct calculation using $S(0, 0) = 1$ and the recursive relation, the following results are obtained in order. Each case (a, b, d, e) has at most 2 eligible neighbors and can be easily derived as in [3]:

a) $S(6, 0) = p^6 \cdot [1 + (1-p)]$
b) $S(5, 1) = p^6 \cdot [1 + 5(1-p)] \ge S(6, 0)$; thus turn `left` at (6, 1), then
c) $S(6, 1) = p \cdot S(5, 1) + p \cdot (1-p) \cdot S(5, 1) + p \cdot (1-p)^2 \cdot S(6, 0)$ [because $S(7, 1) = S(5, 1)$]; thus $S(6, 1) = p^7 \cdot [1 + 6(1-p) + 6(1-p)^2 + (1-p)^3]$
d) $S(5, 2) = p^7 \cdot [1 + 6(1-p) + 14(1-p)^2] \ge S(6, 1)$; thus turn `left` at (6, 2), then
e) $S(6, 2) = p^8 \cdot [1 + 7(1-p) + 21(1-p)^2 + 20(1-p)^3 + 6(1-p)^4 + (1-p)^5]$
f) $S(5, 3) = p^8 \cdot [1 + 7(1-p) + 20(1-p)^2 + 28(1-p)^3]$.

Fact 2: For a 12 × 12 torus, the optimal routing of a message from $(i, j) = (6, 3)$ to (0, 0) depends on p.

Based on Fact 1, $S(5, 3) - S(6, 2) = p^9 \cdot (1-p)^2 \cdot (7 - 9p + p^2)$. The $(9 - \sqrt{53})/2$ is the unique root of $(7 - 9p + p^2)$ in (0, 1), and $(7 - 9p + p^2)|_{p=1} = -1$. Thus,

- $S(5, 3) < S(6, 2)$ when $p > (9 - \sqrt{53})/2$,
- $S(5, 3) > S(6, 2)$ when $p < (9 - \sqrt{53})/2$.

Therefore, the selection of a neighbor (5, 3) or (6, 2) of source (6, 3) depends on p.

Counter-Example: The MSP routing is not optimal for a 12 × 12 torus with $p > (9 - \sqrt{53})/2$.

On a 12 × 12 torus with $p > (9-\sqrt{53})/2$ and with the source $(i, j) = (6, 3)$, the difference between the MSP routing and the optimal SPR is demonstrated in Fig. 2. Because $N = 12$ for a 12 × 12 torus, the source node

$$(i, j) = (6, 3) = \left(\frac{N}{2}, \left\lfloor \frac{N}{4} \right\rfloor\right)$$

is exactly the turning point on column $N/2$. Therefore, according to the MSP routing, $P(5, 3) > P(6, 2)$ and the message at $(i, j) = (6, 3)$ should be forwarded `left` to node $(i-1, j) = (5, 3)$.

On the other hand, according to Fact 2, for a 12 × 12 torus with

$$p > \frac{9 - \sqrt{53}}{2},$$

$$S(5, 3) - S(6, 2) = p^9 \cdot (1-p)^2 \cdot (7 - 9p + p^2) < 0.$$

Thus, the message should be forwarded `down` to node (6, 2) in an optimal SPR. Therefore, the MSP routing is not optimal for a 12 × 12 torus with $p > (9 - \sqrt{53})/2$.

For a larger $N \times N$ torus, $S(i, j)$ can be computed analogously. Similarly to $N = 12$, it is conjectured that the selection of the optimal route at the turning point depends on the value of p for all even N larger than 12. This confirms the conjecture [2] that the optimal policy for the torus seems unlikely to be of a simple closed form.

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