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ON THE CONVERGENCE OF JACOBI AND GAUSS-SEIDEL
ITERATION FOR STEADY-STATE PROBABILITIES OF
FINITE-STATE CONTINUOUS-TIME MARKOV CHAINS

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ABSTRACT

We consider two nonsingular versions of the problem described in the title. For one of these versions, we show by example that neither Jacobi nor Gauss-Seidel iteration is guaranteed to converge; for the other version, we outline a proof that both methods are guaranteed to converge.

1. Introduction

The steady-state probability distribution for a finite-state-space continuous-time Markov process (CTMP) with n states is determined by the solution of

the set of linear equations

$$\mathbf{Q}^T \boldsymbol{\pi} = \mathbf{0}, \quad (1)$$

$$\mathbf{e}^T \boldsymbol{\pi} = 1, \quad (2)$$

where $\mathbf{Q} = [q(i, j)]$ is the (assumed irreducible) $n \times n$ rate matrix ($q(i, i) > 0$, $q(i, j) \leq 0$ when $i \neq j$, and $q(i, i) = -\sum_{j \neq i} q(i, j)$); $\mathbf{0}$ is a column vector

with n zero elements; \mathbf{e} is a column vector with n elements each equal to one; and $\boldsymbol{\pi}$ is the n -component column vector of steady-state probabilities. (We adopt the usual convention that all vectors are column vectors and T denotes transpose. For later convenience, we have defined \mathbf{Q} in (1) as the negative of its usual definition.)

Since one equation in (1) is redundant, the system shown in (1) and (2) can be reduced to the nonsingular system

$$\overline{\mathbf{Q}} \boldsymbol{\pi} = \mathbf{e}_i, \quad (3a)$$

where \mathbf{e}_i is an n -component column vector with $n - 1$ zero elements and the i^{th} element equal to one, and $\overline{\mathbf{Q}}$ is the matrix \mathbf{Q}^T with the i^{th} row replaced by elements each equal to one.

Another modification that changes the singular system of (1) and (2) to a nonsingular system is to arbitrarily set one $\pi_i = 1$ and reduce the set of equations (1) to the $(n - 1) \times (n - 1)$ nonsingular set

$$\hat{\mathbf{Q}} \hat{\boldsymbol{\pi}} = \mathbf{f}, \quad (3b)$$

where $\hat{\boldsymbol{\pi}}$ is an $(n - 1)$ -element vector, $\hat{\mathbf{Q}}$ is a principal submatrix of \mathbf{Q}^T , and \mathbf{f} is the resulting $(n - 1)$ -element right-hand side (no longer $\mathbf{0}$ or \mathbf{e}_i). After solving for the $n - 1$ $\hat{\pi}_j$, these, together with $\pi_i = 1$, can be renormalized.

The difficulty often encountered in employing direct methods to solve either of the nonsingular formulations is that the matrix \mathbf{Q} can be very large when it is generated from certain classes of CTMPs, e.g., those describing closed queueing networks. In these cases, iterative methods such as Jacobi (J) and Gauss-Seidel (GS) are attractive; therefore, questions concerning convergence of these methods become relevant.

According to the conventional wisdom, J and GS are guaranteed to converge for nonsingular versions of this problem; however, there does not seem to be an explicit proof available in the published literature. Therefore, to satisfy our own curiosity, several years ago we constructed such a proof for version (3b) [Cooper, Gross, and Kioussis (1987)]; however, we were unsuccessful in publishing this proof because the referees were unanimous in their opinion that the fact of the convergence was well known and, furthermore,

was too easily provable from the general theory [e.g., Varga (1963), Young (1971), Berman and Plemmons (1979)] to warrant separate publication.

Recently, a comprehensive and scholarly study [Barker (1989)] cited our unpublished paper for the proof of convergence of GS when applied to version (3b). This lends credence to our original motivation for attempting publication, namely, there seems to be no single place in the published literature one can cite for a proof of the convergence of J or GS for version (3b). Furthermore, it does not seem to be widely appreciated that version (3a) and version (3b) are essentially different: When J or GS is applied to version (3a), convergence is *not* guaranteed. (We note in passing also that convergence is not guaranteed when J or GS is applied to the singular version (1) [Kaufman (1983)], but in those cases where convergence does occur, convergence is often much faster than for the nonsingular cases.) Hence, this short note has two objectives: (i) to show (by counterexample) that J and GS are not guaranteed to converge when applied to version (3a), and (ii) to provide for the record (an outline of) a proof that both J and GS are guaranteed to converge when applied to version (3b).

2. Convergence Not Guaranteed For (3a): Counterexample

For our counterexample, we consider a classic machine-repair-with-spares queueing model with a single repairman and 6 machines, one of which is a spare, each machine having an exponential failure-time distribution, with a mean time-to-failure of one, and an exponential repair-time distribution, with a mean time-to-repair of $1/20$. (π_i , $0 \leq i \leq 6$, is the steady-state probability that i machines are in need of repair and, when $i > 0$, the repairman is working on one of them.) The rate matrix \mathbf{Q} is 7×7 and, when the last row of \mathbf{Q}^T is the one replaced, the corresponding matrix $\bar{\mathbf{Q}} = [\bar{q}(i, j)]$ of (3a) is defined by $\bar{q}(0, 0) = 5$, $\bar{q}(i, i) = 26 - i$ when $1 \leq i \leq 5$, $\bar{q}(i-1, i) = -20$ when $1 \leq i \leq 6$, $\bar{q}(1, 0) = -5$, $\bar{q}(i, i-1) = i - 7$ when $2 \leq i \leq 5$, $\bar{q}(6, j) = 1$ when $0 \leq j \leq 6$, and $\bar{q}(i, j) = 0$ otherwise.

It is well known [e.g., Young (1971), Theorem 5.1, p. 77] that a necessary and sufficient condition for convergence of a first-degree linear stationary iterative method is that the spectral radius of the iteration matrix \mathbf{G} be strictly less than one: $S(\mathbf{G}) < 1$. Writing $\bar{\mathbf{Q}} = \mathbf{L} + \mathbf{D} + \mathbf{U}$, where \mathbf{L} is a strict lower-triangular matrix, \mathbf{D} is a diagonal matrix, and \mathbf{U} is a strict upper-triangular matrix, the Jacobi iteration matrix is $\mathbf{G}_1 = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$. A straightforward numerical calculation (using the IMSL eigenvalue package, for example) shows that $1.23 < S(\mathbf{G}_1) < 1.24$; that is, $S(\mathbf{G}_1) > 1$, and hence Jacobi iteration will not converge.

A similar calculation for the eigenvalues for the *GS* iteration matrix $\mathbf{G}_2 = (\mathbf{I} + \mathbf{D}^{-1}\mathbf{L})^{-1}(-\mathbf{D}^{-1}\mathbf{U})$ shows that $1.15 < S(\mathbf{G}_2) < 1.16$; that is, $S(\mathbf{G}_2) > 1$, and hence Gauss-Seidel iteration will not converge, either. Interestingly, if the mean time to repair is increased to $1/10$, then $S(\mathbf{G}_2) < 1 < S(\mathbf{G}_1)$; and when the mean repair time is increased to $3/10$, then both $S(\mathbf{G}_1) < 1$ and $S(\mathbf{G}_2) < 1$.

Thus neither Jacobi iteration nor Gauss-Seidel iteration is guaranteed to converge when applied to version (3a). We now outline a proof that both methods are guaranteed to converge when applied to version (3b).

3. Convergence Guaranteed For (3b): Proof

By assumption, $\mathbf{Q} = [q(i, j)]$ is singular and irreducible (and therefore so is \mathbf{Q}^T), with $q(i, i) > 0$, $q(i, j) \leq 0$ when $i \neq j$, and $q(i, i) = -\sum_{j \neq i} q(i, j)$. It follows (from Gerschgorin's theorem, e.g., Varga (1963), p. 16) that all real eigenvalues of \mathbf{Q} are nonnegative, and similarly for \mathbf{Q}^T (because a matrix and its transpose have the same eigenvalues, e.g., Marlow (1978), p. 127). Therefore, \mathbf{Q}^T is an M-matrix (Berman and Plemmons (1979), condition C_8 of Theorem (4.6), p. 149). $\hat{\mathbf{Q}}$ (defined in (3b)) is a principal submatrix of \mathbf{Q}^T ; therefore $\hat{\mathbf{Q}}$ is a nonsingular M-matrix (Berman and Plemmons (1979), (4) of Theorem (4.16), p. 156). Finally, $S(\mathbf{G}_2) \leq S(\mathbf{G}_1) < 1$ (Berman and Plemmons (1979), (2) of Corollary (5.22), p. 187), and hence both *J* and *GS* converge (e.g., Young (1971), Theorem 5.1, p. 77) when applied to (3b).

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