Reprinted from OPERATIONS RESEARCH
Vol. 20, No. 2, March–April, 1972
Printed in U.S.A.

Queues with Service in Random Order

Grace M. Carter

The Rand Corporation, Santa Monica, California

and

Robert B. Cooper

Georgia Institute of Technology, Atlanta, Georgia

(Received November 24, 1970)

We consider two models, the $GI/M/s$ queue and the $M/G/1$ queue, in which waiting customers are served in random order. For each model we derive expressions for the calculation of the stationary waiting-time distribution function. Our methods differ from those of previous authors in that we do not use transforms, and consequently our results may be better suited for calculation. We illustrate our methods by deriving previously known results for the $M/M/s$ and $M/D/1$ random-service queues, and by making sample calculations for the $M/E_k/1$ random-service queue for various values of the utilization factor and the index k.

WE CONSIDER queues in which, when a server becomes available, the next customer to enter service is chosen at random from among all customers waiting. For this queue discipline of random service, we give methods for calculating the waiting-time distribution function $W(t)$, which specifies, for every $t \geqq 0$, the probability that an arbitrary customer will wait no longer than t for service to begin.

More precisely, we investigate the stationary random-service waiting-time distribution function for two classes of queues, the $GI/M/s$ queue [General Independent interarrival times, Markov (exponential) service times, $s \geqq 1$ servers] and the $M/G/1$ queue [Markov interarrival times (Poisson input), General service times, 1 server]. In each case, the number of waiting positions is infinite. The most commonly studied queue, $M/M/s$, can be considered to be a special case of both these models.

For both the $GI/M/s$ and $M/G/1$ random-service queues, we derive expressions that may be suitable for the calculation of the stationary waiting-time distribution function. Our methods differ from those of previous authors in that we do not use transform techniques. Consequently, our results are not expressed in terms of transforms that are difficult to invert, and hence they may be better suited for calculation.

We illustrate our methods by deriving previously known results for the $M/M/s$ and $M/D/1$ [deterministic (constant) service times] random-service queues, and by making sample calculations for the $M/E_k/1$ [Erlangian of order k] random-service queue for various values of the utilization factor and the index k.

PRELIMINARIES

LET THE RANDOM variable W be the duration of time that an arbitrary customer waits for service to begin, and let $W(t)$ be the corresponding waiting-time distribution function, so that

$$1-W(t)=P\{W>t\}. \qquad (t\geqq 0)$$

As will become clear shortly, it is more convenient to work with the complementary waiting-time distribution function $1-W(t)$ than with $W(t)$ directly.

It follows from the definition of conditional probability that

$$P\{W>t\}=P\{W>0\}P\{W>t|W>0\}. \qquad (1)$$

Equation (1) holds for every queuing system; in particular, it is true for any order of service. The first factor on the right side of (1), $P\{W>0\}$, is the probability that an arbitrary arriving customer finds all s servers busy. Let the random variable N be the number of customers in the system (including those in service) found by an arbitrary arriving customer. Then

$$P\{W>0\}=\sum_{j=0}^{j=\infty} P\{N=s+j\}. \qquad (2)$$

Since the state probabilities on the right-hand side of (2) are (clearly) independent of the order of service, it follows that the probability $P\{W>0\}$ that an arrival must wait for service is also independent of the order of service. Hence (1) shows that all the waiting-time information corresponding to the order of service is contained in the conditional probability $P\{W>t|W>0\}$. Therefore, we shall concentrate our attention on the conditional probability $P\{W>t|W>0\}$, from which the unconditional waiting-time distribution function can easily be calculated by using (1). Also, as we shall see, the conditional probability $P\{W>t|W>0\}$ requires fewer parameters for its specification than does the unconditional probability $P\{W>t\}$.

It follows from the law of total probability that

$$P\{W>t\}=\sum_{j=0}^{j=\infty} P\{N=s+j\}P\{W>t|N=s+j\}. \qquad (3)$$

Like (1), (3) holds for queuing systems in general. Also, all the waiting-time information corresponding to the order of service is contained in the conditional probabilities $P\{W>t|N=s+j\}$, $j=0, 1, \cdots$.

Although (1), (2), and (3) are correct for any queuing system, it does not follow that they are equally useful in the analysis of any queuing system. It happens that these equations are directly applicable to the analysis of the $GI/M/s$ queue, but that the analysis of the $M/G/1$ queue requires further consideration.

In the case of queues with Poisson arrivals at rate λ, the probability $p_i(x)$ that exactly i customers arrive in any interval of length x is, by definition, the Poisson probability

$$p_i(x)=[(ax)^i/i!]e^{-ax}, \qquad (i=0, 1, \cdots) \quad (4)$$

where $a=\lambda$.

We shall also use the fact that, for s-server queues with exponential service times with mean length μ^{-1}, the conditional probability that exactly i customers complete service in an interval of length x, given that all s servers are continuously busy throughout this interval, is also given by the Poisson probability (4), with

$a = s\mu$. (Note also that, when the service times are exponential, the conditional probability $P\{W>t|W>0\}$ for the s-server queue can be obtained from that for the corresponding single-server queue by replacing μ by $s\mu$.)

THE $M/M/s$ RANDOM-SERVICE QUEUE

THE $M/M/s$ RANDOM-service queue has been studied by several authors. In this section we review some of their results, and in the process we discuss some concepts needed for our analysis of the $GI/M/s$ and $M/G/1$ random-service queues.

Let λ be the customer arrival rate, and μ^{-1} be the mean service time, with utilization factor $\rho = \lambda/s\mu$. It is well known that the probability $P\{N=s+j\}$ that an arriving customer finds all s servers busy and $j \geqq 0$ other customers waiting for service is given by the geometric distribution,

$$P\{N=s+j\} = \rho^j P\{N=s\}. \qquad (j=0, 1, \cdots) \quad (5)$$

Then, equating the right-hand sides of (1) and (3), and making use of equations (2) and (5), we have, for the $M/M/s$ queue in equilibrium,

$$P\{W>t|W>0\} = (1-\rho) \sum_{j=0}^{j=\infty} \rho^j P\{W>t|N=s+j\}. \quad (6)$$

Equation (6) describes the waiting times in the stationary $M/M/s$ queue with any order of service.

For example, suppose that customers are served in their arrival order. Consider a customer (called the *test customer*) who finds all s servers busy and $j \geqq 0$ other customers waiting for service when he arrives. The test customer will wait in excess of t for his service to begin if and only if there are less than $(j+1)$ service completions within this interval of length t. Therefore, for queues with exponential service times and service in order of arrival,

$$P\{W>t|N=s+j\} = \sum_{i=0}^{i=j} p_i(t), \quad (7)$$

where $p_i(t)$ is given by the Poisson probabilities (4), with $a = s\mu$. Insertion of (7) into (6) yields the well known result for the stationary $M/M/s$ queue with service in order of arrival:

$$P\{W>t|W>0\} = e^{-(1-\rho)s\mu t}. \quad (8)$$

We now turn to the case in point, service in random order. In contrast to the case of service in arrival order, it is not possible to write down an expression for the conditional probability $P\{W>t|N=s+j\}$ directly from elementary considerations. For convenience, we define $W_j(t) = P\{W>t|N=s+j\}$ for queues with random service.

By considering all possible transitions during a time interval $(t, t+h)$ as $h \to 0$, the following set of differential-difference equations can be derived:

$$(d/dt)W_j(t) = \lambda W_{j+1}(t) - (\lambda + s\mu)W_j(t) + [j/(j+1)]s\mu W_{j-1}(t). \quad [j=0, 1, \cdots; W_j(0)=1] \quad (9)$$

Equation (9) was derived independently in 1946 by VAULOT[16] and PALM.[8]

(According to SYSKI,[12] Palm's work was done in 1938 but was not published until 1946.) POLLACZEK,[9] also in 1946, obtained a closed-form solution for the conditional waiting-time probability $P\{W>t|W>0\}$. Summaries of these and other studies on the $M/M/s$ random-service queue are given in RIORDAN[11] and Syski.[12]

In 1953 Riordan[10] obtained the Maclaurin-series representation of the conditional probability $P\{W>t|W>0\}$ as follows: Let

$$W_j^{(\nu)}=(d^\nu/dt^\nu)W_j(t)|_{t=0}, \qquad [j=0,1,\cdots;\ \nu=0,1,\cdots;\ W_j^{(0)}=1] \tag{10}$$

and assume that $W_j(t)$ has the Maclaurin-series representation

$$W_j(t)=\sum_{\nu=0}^{\nu=\infty}(t^\nu/\nu!)W_j^{(\nu)}. \qquad (j=0,1,\cdots) \tag{11}$$

Then (6) can be written

$$P\{W>t|W>0\}=1+(1-\rho)\sum_{\nu=1}^{\nu=\infty}(t^\nu/\nu!)\sum_{j=0}^{j=\infty}\rho^jW_j^{(\nu)}. \tag{12}$$

The derivatives $\{W_j^{(\nu)}\}$ appearing on the right-hand side of (12) can be determined from the basic recurrence (9). Riordan evaluates the coefficients of t, t^2, and t^3 explicitly, giving

$$\begin{aligned}P\{W>t|W>0\}=1-s\mu t[(1-\rho)/\rho]\ln[1/(1-\rho)]\\+[(s\mu t)^2/2!](1-\rho)\{2-[(1-\rho)/\rho]\ln[1/(1-\rho)]\}\\-[(s\mu t)^3/3!](1-\rho)\{1+3\rho-(1-\rho)\ln[1/(1-\rho)]-\textstyle\sum_{j=1}^{j=\infty}(\rho^j/j^2)\}+-\cdots.\end{aligned} \tag{13}$$

However, Riordan mentions this result only for the sake of "completeness"; his main result is an approximation of $1-W(t)$ as a weighted sum of exponential terms. Riordan makes no calculations from this series (computers have come a long way since 1953) and asserts only that the series converges for "small values" of $s\mu t$. As we shall discuss, recent results of P. J. BURKE show that the series (13) converges for all $s\mu t\geqq 0$.

Note that equation (13) is a power series in $s\mu t$; we might just as well have taken $(s\mu)^{-1}$ as the unit of time (as Riordan does). This shows that the conditional waiting-time distribution function depends on only two parameters, ρ and the product $s\mu t$. In contrast, the unconditional waiting-time distribution function requires three parameters, ρ, μt, and s, for its complete characterization.

In the present study we give, among other results, heuristic derivations of the Maclaurin-series representations of the conditional probability $P\{W>t|W>0\}$ for the $GI/M/s$ and $M/G/1$ random-service queues. We show that specialization of our results (for Poisson input in the first case and exponential service times in the second case) leads to (13), and we give results of sample calculations based on these representations.

THE $GI/M/s$ RANDOM-SERVICE QUEUE

THE $GI/M/s$ random-service queue has been considered previously by LEGALL[7] and TAKÁCS.[14] LeGall gives an expression for the characteristic function of the waiting-time distribution function, and Takács gives the corresponding Laplace-

Stieltjes transform. Their results are quite formidable, and reduction to practice (through explicit or numerical inversion) does not appear simple.

We first give a heuristic derivation of the Maclaurin-series expansion of the conditional waiting-time distribution function, and show that our results include (13) as a special case. We then summarize another method, proposed by P. J. Burke,[2] from which it follows that the Maclaurin series (13) converges for all $s\mu t \geqq 0$. Neither of these methods uses transform techniques. Both methods will be used in the next section to develop analogous results for the $M/G/1$ queue.

Our starting point is equations (1)–(3), which are valid for queuing systems in general. Corresponding to (5) for the $GI/M/s$ queue is

$$P\{N=s+j\}=\omega^j P\{N=s\}, \qquad (j=0, 1, \cdots) \quad (14)$$

where ω is the only root in (0, 1) of the equation

$$\omega=\gamma[s\mu(1-\omega)], \quad (15)$$

and $\gamma(z)$ is the Laplace-Stieltjes transform of the interarrival-time distribution function $G(\xi)$,

$$\gamma(z)=\int_0^\infty e^{-z\xi}dG(\xi). \quad (16)$$

The complete set of state probabilities for the $GI/M/s$ queue are given, for example, in Riordan,[11] Syski,[12] or Takács.[13] The root ω can easily be calculated according to the iteration scheme,

$$\omega_{i+1}=\gamma[s\mu(1-\omega_i)]. \qquad (i=0, 1, \cdots; 0\leqq\omega_0<1) \quad (17)$$

Equating the right-hand sides of (1) and (3), and making use of (2) and (14), we have, for the $GI/M/s$ queue in equilibrium,

$$P\{W>t|W>0\}=(1-\omega)\textstyle\sum_{j=0}^{j=\infty}\omega^j P\{W>t|N=s+j\}. \quad (18)$$

Equation (18) is similar in form to (6), and reduces to it when $G(\xi)=1-e^{-\lambda\xi}(\xi\geqq 0)$. It remains to calculate the conditional probabilities $P\{W>t|N=s+j\}=W_j(t)$ $(j=0, 1, \cdots)$ for the $GI/M/s$ random-service queue. The analysis leading to (9) for the $M/M/s$ queue was based on the fact that the $M/M/s$ queue can be described as a Markov process. Since we are no longer assuming Poisson input, the $GI/M/s$ queue cannot be described by a Markov process, but can be described by an imbedded (at the customer-arrival epochs) Markov chain. That is, the methods used by Vaulot, Palm, and Pollaczek to study the $M/M/s$ random-service queue are not applicable to the study of the $GI/M/s$ random-service queue.

We give two methods for determining $W_j(t)$, $j=0, 1, \cdots$, the *Maclaurin-series method* and Burke's *additional-conditioning-variable method.*

The Maclaurin-Series Method

Consider a test customer who, upon arrival at time T_c, say, finds all s servers busy and $j\geqq 0$ other customers waiting for service. We wish to calculate the probability $W_j(t)$ that the test customer waits in excess of t for his service to begin. Consider the two events: (a) the next customer arrives after time T_c+t, or (b) the next customer arrives prior to time T_c+t.

In case (a), the test customer will wait more than t for service to begin if and only if he is one of the $i(1\leqq i\leqq j+1)$ customers still waiting for service at time T_c+t. (If $i=0$, the test customer will necessarily have begun service.) Since service times are assumed to be independently exponentially distributed with common mean μ^{-1}, the probability $p_i(x)$ that exactly i customers complete service in an interval of length x, given that all s servers are continuously busy throughout this interval, is the Poisson probability (4) with $a=s\mu$. Let $T_{c'}$ be the next arrival epoch after T_c.

If $T_{c'}-T_c>t$, then the (conditional) probability that the test customer waits in excess of t for service to commence is

$$P\{W>t|N=s+j,\ T_{c'}-T_c>t\}=\sum_{i=1}^{j+1}[i/(j+1)]p_{j+1-i}(t), \qquad (19)$$

where the factor $i/(j+1)$ is the probability that the test customer is not among the $j+1-i$ customers selected (according to the random selection procedure) to begin service during the interval $(T_c,\ T_c+t)$. Since the interarrival-time distribution function is $G(x)$, and since T_c is an arrival epoch, event (a) occurs with probability $1-G(t)$.

Now consider event (b); that is, suppose that the next customer arrives at time $T_{c'}=T_c+\xi$, where $\xi\leqq t$. The probability that the test customer will be among the remaining i waiting customers is $[i/(j+1)]p_{j+1-i}(\xi)$. The test customer will now experience a total wait in excess of t for service to begin if and only if he suffers an additional delay exceeding length $t-\xi$. But, since waiting customers are selected for service in random order, the probability that the test customer's additional waiting time will exceed $t-\xi$ (if he has not yet begun service) is the same as the probability that a new arrival waits in excess of $t-\xi$ for service to begin. This latter probability is $W_i(t-\xi)$. Thus, if the next customer arrives at time $T_{c'}=T_c+\xi$, where $\xi\leqq t$, then the test customer's waiting time will exceed t with probability

$$P\{W>t|N=s+j,\ T_{c'}-T_c=\xi\leqq t\}=\sum_{i=1}^{j+1}[i/(j+1)]p_{j+1-i}(\xi)W_i(t-\xi). \qquad (20)$$

Finally, the probability that the next arrival epoch $T_{c'}$ will occur in an infinitesimal interval about the point $T_c+\xi$ is $dG(\xi)$.

Therefore, combining events (a) and (b), we have the following recurrence for the conditional probability $W_j(t)$ that the test customer waits in excess of t for service to begin, given that on arrival he finds all s servers busy and $j\geqq 0$ other customers waiting for service:

$$\begin{aligned} W_j(t)=&[1-G(t)]\sum_{i=1}^{j+1}[i/(j+1)]p_{j+1-i}(t)\\ &+\sum_{i=1}^{j+1}[i/(j+1)]\int_0^t p_{j+1-i}(\xi)W_i(t-\xi)\,dG(\xi), \qquad (j=0,1,\cdots) \end{aligned} \qquad (21)$$

where $G(x)$ is the interarrival-time distribution function and $p_i(x)$ is given by equation (4) with $a=s\mu$.

We now assume that $W_j(t)$ for the $GI/M/s$ queue has the Maclaurin-series representation

$$W_j(t)=\sum_{\nu=0}^{\nu=\infty}(t^\nu/\nu!)W_j^{(\nu)}. \qquad (j=0,1,\cdots;\ W_j^{(0)}=1) \quad (22)$$

Equation (22) is the same as (11), except that (22) refers to the $GI/M/s$ queue, while (11) refers to the $M/M/s$ queue. Using the same reasoning that led from (11) to (12) for the $M/M/s$ queue, we have the analogous equation corresponding to the $GI/M/s$ queue:

$$P\{W>t|W>0\}=1+(1-\omega)\sum_{\nu=1}^{\nu=\infty}(t^{\nu}/\nu!)\sum_{j=0}^{j=\infty}\omega^{j}W_{j}^{(\nu)}. \tag{23}$$

It remains to determine the derivatives $\{W_j^{(\nu)}\}$ appearing on the right-hand side of (23) from the basic recurrence (21).

Formally differentiating ν times on both sides of equation (21) we have

$$\begin{aligned}(d^{\nu}/dt^{\nu})W_j(t)=(d^{\nu}/dt^{\nu})&\left\{[1-G(t)]\sum_{i=1}^{j+1}[i/(j+1)]p_{j+1-i}(t)\right\}\\&+\sum_{i=1}^{j+1}[i/(j+1)]\int_0^t p_{j+1-i}(\xi)g(\xi)(\partial^{\nu}/\partial t^{\nu})W_i(t-\xi)\,d\xi\\&+\sum_{i=1}^{j+1}[i/(j+1)]\sum_{k=0}^{\nu-1}(d^k/dt^k)[p_{j+1-i}(t)g(t)]W_i^{(\nu-1-k)},\\&\qquad(j=0,1,\cdots;\ \nu=1,2,\cdots)\end{aligned} \tag{24}$$

where $g(\xi)$ is the interarrival-time density function,

$$g(\xi)=(d/d\xi)G(\xi). \tag{25}$$

For convenience, set

$$a_{j+1}(t)=[1-G(t)]\sum_{i=1}^{j+1}[i/(j+1)]p_{j+1-i}(t) \tag{26}$$

and

$$b_{j+1-i}(t)=g(t)p_{j+1-i}(t). \tag{27}$$

Now set $t=0$ in equation (24). The integral on the right-hand side vanishes and we have

$$W_j^{(\nu)}=a_{j+1}^{(\nu)}+\sum_{i=1}^{j+1}[i/(j+1)]\sum_{k=0}^{\nu-1}b_{j+1-i}^{(k)}W_i^{(\nu-1-k)}. \qquad (j=0,1,\cdots;\ \nu=1,2,\cdots) \tag{28}$$

The recurrence (28) permits evaluation of the sum on the right-hand side of (23); the problem is solved if the series converges and if the assumed derivatives exist. If, in addition, the terms of the series are easy to calculate, the solution is also useful.

We note in passing that this analysis was motivated by a study[3, 4] of the Bell System's No. 101 Electronic Switching System. In this system, k groups of s dial-tone machines (servers) provide dial tone in random order to waiting calls. Each group of s dial-tone machines provides service for every kth call so that, if calls occur according to a Poisson process, the distribution function of the interarrival times of calls at any group of s dial-tone machines is the k-order Erlangian

$$G(t)=1-\sum_{j=0}^{k-1}[(\lambda t)^{j}/j!]e^{-\lambda t}. \qquad (k=1,2,\cdots) \tag{29}$$

Dial-tone-delay curves for engineering this system were calculated according to the above algorithm.

In particular, it is easy to verify that, when $k=1$ in equation (29), direct calculation by means of the recurrence (28) gives, for the first three terms of the expansion (23), the previous result (13) of Riordan for the $M/M/s$ random-service queue.

The Additional-Conditioning-Variable Method

The Maclaurin-series method assumes the existence of the Maclaurin-series representation of $P\{W>t|W>0\}$, and gives no information as to the range of convergence of this series, if indeed it does exist. In addition, the Maclaurin-series method requires that the interarrival-time distribution function $G(x)$ possess a density; this constraint precludes analysis of the important case of constant inter-arrival times.

Burke[2] has proposed the introduction of an additional 'conditioning variable,' which meets some of these objections. According to Burke, this method yields results that appear to be well suited for computation on a digital computer. For the particular case of Poisson input, Burke's method yields a power-series representation for $P\{W>t|W>0\}$ that is easily shown to be convergent for all $s\mu t\geqq 0$. It follows from the uniqueness property of power-series representations that the Maclaurin series (13) exists and converges for all $s\mu t\geqq 0$. It appears that further investigation of the $GI/M/s$ random-service queue by Burke's additional-conditioning-variable method might similarly throw light on the existence and convergence properties of the Maclaurin-series representation of $P\{W>t|W>0\}$ for other choices of the input process. Then the choice of method to be used for computation could be based solely on other (computational) considerations.

In this section we briefly describe Burke's additional-conditioning-variable method, which we shall later extend to the analysis of the $M/G/1$ random-service queue.

The Maclaurin-series method was based on obtaining an expression for $W_j(t)=P\{W>t|N=s+j\}$ for $j=0, 1, \cdots$, where W is the waiting time of the test customer, and N is the number of customers present in the system just prior to the test customer's arrival epoch T_c. Burke considers the additional random variable $X(t)$, defined as the number of customers who arrive in $(T_c, T_c+t]$, with $t>0$, and $X(0)=0$ with probability one. Instead of calculating $W_j(t)$, Burke's basic calculation is that of $W_{j,k}(t)$, where we define

$$W_{j,k}(t)=P\{W>t|N=s+j,\ X(t)=k\}. \qquad (j=0, 1, \cdots; k=0, 1, \cdots)$$

We have that

$$P\{X(t)=k\}=G^{*k}(t)-G^{*(k+1)}(t), \qquad (k=0, 1, \cdots; G^{*0}=1; t\geqq 0) \quad (30)$$

where $G^{*k}(t)$ is the k-fold convolution of the interarrival-time distribution function $G(t)$ with itself. Also, from the law of total probability,

$$P\{W>t|N=s+j\}=\textstyle\sum_{k=0}^{k=\infty} P\{X(t)=k\}P\{W>t|N=s+j,\ X(t)=k\}. \quad (31)$$

Since the conditional waiting-time distribution function is determined by (18), it follows from (31) that it is now sufficient to determine the conditional probabilities $\{W_{j,k}(t)\}$.

First note that $X(t)=0$ if and only if $T_{c'}-T_c>t$. It follows from (19) and the definition of $W_{j,k}(t)$ that

$$W_{j,0}(t)=\textstyle\sum_{i=1}^{j+1} [i/(j+1)]p_{j+1-i}(t), \qquad (j=0, 1, \cdots) \quad (32)$$

where $p_i(x)$ is given by equation (4) with $a=s\mu$.

For $k>0$ we define

$$G(\xi|k, t) = P\{T_{c'} - T_c \leqq \xi | X(t) = k\}, \tag{33}$$

which is the conditional distribution function of the elapsed time $T_{c'} - T_c$ between the arrival epoch T_c of the test customer and the next arrival epoch $T_{c'}$, given that k arrivals occur in $(T_c, T_c + t]$. It follows from the definition of a conditional probability density function that

$$d_\xi G(\xi|k, t) = P\{X(t-\xi) = k-1\} dG(\xi) / P\{X(t) = k\}. \qquad (k=1, 2, \cdots) \tag{34}$$

Then, reasoning in a manner similar to that leading to equation (21), we have the following general recurrence for $k \geqq 1$:

$$W_{j,k}(t) = \sum_{i=1}^{j+1} [i/(j+1)] \int_0^t p_{j+1-i}(\xi) W_{i,k-1}(t-\xi)\, d_\xi G(\xi|k, t). \tag{35}$$
$$(j=0, 1, \cdots; k=1, 2, \cdots)$$

Thus, by use of (32) and (35), one can compute $P\{W > t | W > 0\}$ to any desired degree of accuracy. Implementation of this algorithm is discussed by Burke.[2] Burke illustrates his method for the case of constant interarrival times, for which the Maclaurin-series method is inapplicable, and the case of exponential interarrival times, for which he obtains an infinite series whose convergence is assured, and from which we can infer the convergence of the Maclaurin series (13).

THE $M/G/1$ RANDOM-SERVICE QUEUE

WE NOW TURN TO the stationary $M/G/1$ queue with service of waiting customers in random order. This model has been studied by LeGall[7] and KINGMAN[6] (*see* also Takács[15]). LeGall and Kingman used transform methods, and numerical implementation of their results does not appear simple.

The special case $M/D/1$ has also been studied by Burke,[1] who obtained an expression suitable for calculating the conditional waiting-time distribution function. Burke's analysis utilizes certain simplifying properties of the constant-service-time distribution that do not hold in the general case.

Our approach is to extend the two methods (Maclaurin-series and additional-conditioning-variable) used in the analysis of the $GI/M/s$ random-service queue to the analysis of the $M/G/1$ random-service queue. As before, neither method uses transform techniques, and hence our results may be more suitable for calculation than LeGall's or Kingman's.

As illustrations of our methods, we derive Riordan's expansion (13) for the $M/M/1$ random-service queue by specializing the Maclaurin-series analysis; and we derive Burke's results for the $M/D/1$ random-service queue by specializing the additional-conditioning-variable analysis.

Suppose that an arbitrary customer (the test customer) arrives at time T_c and finds $j \geqq 1$ other customers in the system (either in service or waiting for service). Then the length of time that the test customer waits for service to begin is the sum of (a) the time between the test customer's arrival epoch T_c and the first subsequent departure (service completion) epoch T_1, and (b) the time between T_1 and the instant at which the test customer commences service.

The analysis of the total waiting time is complicated by two facts. First, the time intervals (a) and (b) are not, in general, statistically independent. Second, any customer in service at epoch T_c when the test customer arrives does not, in general, have the same service-time distribution function as does an arbitrary customer.

In the special case of constant service times these difficulties disappear. When the service times are constant, the delay suffered by a blocked customer between the instant of his arrival and the first subsequent departure epoch is independent of the remaining delay beyond this epoch; and the distribution of the service interval containing T_c is the same as that of an arbitrary service interval (that is, constant). These simplifying properties were noted explicitly by Burke in his analysis of the $M/D/1$ random-service queue.

As before, we let W be the duration of the test customer's wait for service to begin, and we calculate the conditional probability $P\{W>t|W>0\}$. We denote by T_c the test customer's arrival epoch, and by T_1 the first departure epoch subsequent to T_c. We also define $N(T)$ as the number of customers in the system (including service) at any time T, and $W(T)$ as the remaining waiting time for service to commence for a customer who is waiting at time T. Then we may write

$$P\{W>t|W>0\}=P\{T_1-T_c>t|W>0\}$$
$$+\sum_{j=2}^{j=\infty}[(j-1)/j]\int_0^t P\{W(T_1+0)>t-\xi|W>0,\,N(T_1+0)=j\} \tag{36}$$
$$\cdot d_\xi P\{N(T_1+0)=j,\,T_1-T_c\leqq\xi|W>0\}.$$

As already noted, the service time of a customer in service at epoch T_c when the test customer arrives does not, in general, have the same distribution function as does an arbitrary service time. Let $H(x)$ be the service-time distribution function for an arbitrary customer, with mean τ; and let $\hat{H}(x)$ be the distribution function of the length of the service interval during which the test customer arrives. (Since we are calculating the conditional probability $P\{W>t|W>0\}$, the test customer is assumed to arrive during some customer's service interval.) Then it is known from renewal theory (*see*, for example, Chapter XI of reference 5) that, if the queue is in equilibrium, the distribution function $\hat{H}(x)$ of the length of the service interval containing the arrival epoch T_c and the distribution function $H(x)$ of the length of an arbitrary service interval are related as follows:

$$d\hat{H}(x)=(1/\tau)x\,dH(x). \tag{37}$$

Likewise, it is well known that the distribution function $\tilde{H}(x)$ of the length of the remainder of this service interval, from T_c to T_1,

$$\tilde{H}(x)=P\{T_1-T_c\leqq x|W>0\}$$

is given by

$$\tilde{H}(x)=(1/\tau)\int_0^x [1-H(\xi)]\,d\xi. \tag{38}$$

Let us define

$$Q_j(\xi)=P\{N(T_1+0)=j,\,T_1-T_c\leqq\xi|W>0\}, \qquad (j=0,1,\cdots)$$

and let $\{\Pi_j^*\}$ be the stationary distribution of the number of customers in the system (waiting or in service) just after a service completion epoch. (The distribution $\{\Pi_j^*\}$ is known, in principle, for any stationary $M/G/1$ queue; *see*, for example, references 11, 12, or 13.)

Consider the service interval containing the test customer's arrival epoch T_c. The probability that this interval has length between x and $x+dx$ is $d\hat{H}(x)$, given by (37). The probability that exactly i customers arrive during such an interval of length x is, by assumption, the Poisson probability (4) with $a=\lambda$, where λ is the arrival rate. Since the arrival process is Poisson, then for fixed x the length of the interval (T_c, T_1) is uniformly distributed over $(0, x)$. It follows that $Q_j(\xi)$ has density $(d/d\xi)Q_j(\xi)=Q_j'(\xi)$ given by

$$Q_j'(\xi)=\int_\xi^\infty [\Pi_0^* p_{j-1}(x)+\sum_{i=1}^{i=j} \Pi_i^* p_{j-i}(x)]\, d\hat{H}(x)/x, \qquad (j=1, 2, \cdots) \quad (39)$$

which, in view of (37), can be written

$$Q_j'(\xi)=(1/\tau)\int_\xi^\infty [\Pi_0^* p_{j-1}(x)+\sum_{i=1}^{i=j} \Pi_i^* p_{j-i}(x)]\, dH(x). \qquad (j=1, 2, \cdots) \quad (40)$$

Finally, let us define

$$\check{W}_j(x)=P\{W(T_1+0)>x|W>0, N(T_1+0)=j\}. \qquad (j=2, 3, \cdots)$$

Then (36) can be written

$$P\{W>t|W>0\}=1-\tilde{H}(t)+\sum_{j=2}^{j=\infty} [(j-1)/j]\int_0^t \check{W}_j(t-\xi)Q_j'(\xi)\, d\xi, \quad (41)$$

where $\tilde{H}(x)$ is given by (38), $Q_j'(\xi)$ is given by (40), and $\check{W}_j(x)$, $j=2, 3, \cdots$, remains to be determined.

We now give two methods for calculating $\check{W}_j(x)$, $j=2, 3, \cdots$, and $P\{W>t|W>0\}$. The first method, which has two variants, is based on the Maclaurin-series expansion. The second method is based on the use of an additional conditioning variable.

The Maclaurin-Series Method

We have the following set of integral equations for the functions $\{\check{W}_j(x)\}$:

$$\check{W}_j(x)=1-H(x)+\sum_{i=0}^{i=\infty} [(j+i-2)/(j+i-1)]\int_0^x p_i(\xi)\check{W}_{j+i-1}(x-\xi)dH(\xi). \quad (42)$$

$$(j=2, 3, \cdots)$$

Equation (42) for the $M/G/1$ random-service queue corresponds to (21) for the $GI/M/s$ random-service queue.

We now assume that $\check{W}_j(x)$ has the Maclaurin-series representation

$$\check{W}_j(x)=\sum_{\nu=0}^{\nu=\infty} (x^\nu/\nu!)\check{W}_j^{(\nu)}. \qquad (j=2, 3, \cdots; \check{W}_j^{(0)}=1) \quad (43)$$

Repeated formal differentiation of (42) yields

$$\frac{d^\nu}{dx^\nu}\check{W}_j(x)=-\frac{d^\nu}{dx^\nu}H(x)+\sum_{i=0}^{\infty}\frac{j+i-2}{j+i-1}\int_0^x p_i(\xi)h(\xi)\frac{\partial^\nu}{\partial x^\nu}\check{W}_{j+i-1}(x-\xi)\,d\xi$$
$$+\sum_{i=0}^{\infty}\frac{j+i-2}{j+i-1}\sum_{k=0}^{\nu-1}p_i(x)h(x)\check{W}_{j+i-1}^{(\nu-i-k)},\qquad (j=2,3,\cdots;\nu=1,2,\cdots)\tag{44}$$

where $h(x)$ is the service-time density function,

$$h(x)=(d/dx)H(x).\tag{45}$$

For convenience, set

$$\check{b}_i(x)=h(x)p_i(x).\tag{46}$$

Now set $x=0$ in equation (44). The integral on the right-hand side vanishes, and, since $\check{b}_i^{(k)}=0$ when $i>k$, we have

$$\check{W}_j^{(\nu)}=-H^{(\nu)}+\sum_{i=0}^{\nu-1}[(j+i-2)/(j+i-1)]\sum_{k=0}^{\nu-1}\check{b}_i^{(k)}\check{W}_{j+i-1}^{(\nu-i-k)}.\qquad (j=2,3,\cdots;\nu=1,2,\cdots)\tag{47}$$

Equation (47) permits evaluation of the sum on the right-hand side of (43). In principle, the required function $P\{W>t|W>0\}$ can now be calculated from (43) and (41) (assuming, of course, convergence of series and existence of derivatives where required).

The calculation of $P\{W>t|W>0\}$, as indicated above, requires the evaluation of the integrals on the right-hand sides of equations (40) and (41). In some cases these integrations may be performed formally. When formal integration is difficult, numerical techniques can be used. In such cases, however, the computational difficulties may be prohibitive. The following variant of the method avoids direct integration, although new numerical problems may be introduced.

Along with the assumption (43), we assume the Maclaurin representation of $P\{W>t|W>0\}$. For notational convenience, let $F(t)=P\{W>t|W>0\}$. Then we assume the representation

$$F(t)=\sum_{\nu=0}^{\nu=\infty}(t^\nu/\nu!)F^{(\nu)}.\qquad (F^{(0)}=1)\tag{48}$$

Evaluation of (48) requires calculation of the derivatives $\{F^{(\nu)}\}$. To this end, formal differentiation ν times of equation (41) yields

$$\frac{d^\nu}{dt^\nu}F(t)=-\frac{d^\nu}{dt^\nu}\tilde{H}(t)+\sum_{j=2}^{\infty}\frac{j-1}{j}\int_0^t Q_j'(\xi)\frac{\partial^\nu}{\partial t^\nu}\check{W}_j(t-\xi)\,d\xi$$
$$+\sum_{j=2}^{\infty}\frac{j-1}{j}\sum_{k=0}^{\nu-1}\frac{d^k}{dt^k}Q_j'(t)\check{W}_j^{(\nu-1-k)}.\qquad (\nu=1,2,\cdots)\tag{49}$$

Now set $t=0$ in (49). We obtain the recurrence

$$F^{(\nu)}=-\tilde{H}^{(\nu)}+\sum_{j=2}^{j=\infty}[(j-1)/j]\sum_{k=0}^{\nu-1}Q_j^{(k+1)}\check{W}_j^{(\nu-1-k)}.\qquad (\nu=1,2,\cdots)\tag{50}$$

It follows from equation (40) that

$$Q_j^{(1)}=Q_j'(0)=(1/\tau)\Pi_{j-1}^*,\qquad (j=1,2,\cdots)\tag{51}$$

and

$$Q_j^{(k+1)}=-\frac{1}{\tau}\sum_{m=0}^{k-1}\binom{k-1}{m}H^{(k-m)}\left(\Pi_0{}^*p_{j-1}^{(m)}+\sum_{i=1}^{j}\Pi_i{}^*p_{j-i}^{(m)}\right).\qquad (k=1,2,\cdots;j=1,2,\cdots)\tag{52}$$

Thus, (47), (50), (51), and (52) permit formal calculation of the Maclaurin-series representation (48) of the required conditional probability $P\{W>t|W>0\} = F(t)$. This method does not require the evaluation of any integrals. Table I gives, for the $M/E_k/1$ random-service queue, sample calculations based on this algorithm.

As a further example, we will use the second variant of the Maclaurin-series method to calculate $P\{W>t|W>0\}$ for the $M/M/1$ random-service queue.

The service-time distribution function is the negative exponential,

$$H(x) = 1 - e^{-x/\tau} \tag{53}$$

and the distribution $\{\Pi_j^*\}$ of the number of customers in the system just after an arbitrary service completion epoch is

$$\Pi_j^* = (1-\rho)\rho^j, \qquad (j=0, 1, \cdots) \tag{54}$$

TABLE I

SAMPLE CALCULATIONS BY THE MACLAURIN-SERIES METHOD FOR $P\{W>t|W>0\}$ FOR THE $M/E_k/1$ RANDOM-SERVICE QUEUE

t	$\rho=0.3$			$\rho=0.8$		
	$k=1$	$k=2$	$k=3$	$k=1$	$k=2$	$k=3$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	0.8487	0.8364	0.8336	0.9254	0.9166	0.9137
0.4	0.7237	0.6889	0.6766	0.8609	0.8395	0.8312
0.6	0.6199	0.5365	0.5394	0.8045	0.7709	0.7565
0.8	0.5331	0.4599	0.4261	0.7547	0.7107	0.6908
1.0	0.4602	0.3758	0.3361	0.7102	0.6578	0.6338
1.5	0.3236	0.2305	0.1894	0.617	0.551	0.521
2.0	0.2319	0.1455	0.112	0.544	0.470	0.440
2.5	0.1689	0.0945	0.072	0.483	0.407	
3.0	0.1248	0.063		0.433	0.358	
3.5	0.093	0.043		0.39	0.33	
4.0	0.070			0.35		
4.5	0.054			0.33		

Note: t is measured in units of mean service time.

where $\rho=\lambda\tau$. From (38) it follows that the distribution function $\tilde{H}(x)$ of the length of the remainder of the service interval measured from the test customer's arrival epoch is the same as the service-time distribution function,

$$\tilde{H}(x) = 1 - e^{-x/\tau}, \tag{55}$$

a result that could have been anticipated in light of the Markov property of the negative-exponential distribution function.

For $\nu = 1$ equation (50) becomes

$$F^{(1)} = -(1/\tau) + \sum_{j=2}^{j=\infty} [(j-1)/j]Q_j^{(1)}. \tag{56}$$

From (51) we have

$$Q_j^{(1)} = (1/\tau)(1-\rho)\rho^{j-1}, \qquad (j=1, 2, \cdots) \tag{57}$$

so that equation (56) can be written

$$F^{(1)}=-(1/\tau)[(1-\rho)/\rho]\ln[1/(1-\rho)]. \tag{58}$$

For $\nu=2$, (50) becomes

$$F^{(2)}=-(1/\tau^2)+\sum_{j=2}^{j=\infty}[(j-1)/j](Q_j^{(1)}\check{W}_j^{(1)}+Q_j^{(2)}). \tag{59}$$

It follows from equation (52) that

$$Q_j^{(2)}=-(1/\tau)H^{(1)}[\Pi_0{}^*p_{j-1}(0)+\sum_{i=1}^{i=j}\Pi_i{}^*p_{j-i}(0)], \tag{60}$$

which reduces to

$$Q_j^{(2)}=-(1/\tau^2)(1-\rho)\rho^j. \qquad (j=1,2,\cdots). \tag{61}$$

Finally, from (47) it follows that

$$\check{W}_j^{(1)}=-(1/\tau)+[(j-2)/(j-1)](1/\tau). \tag{62}$$

Substitution of (57), (61), and (62) into (59) yields, after simplification,

$$F^{(2)}=(1/\tau^2)(1-\rho)\{2-[(1-\rho)/\rho]\ln[1/(1-\rho)]\}. \tag{63}$$

Equations (58) and (63) give the coefficients of the first two nontrivial terms in the Maclaurin expansion (48), in agreement with Riordan's expansion (13) with $s=1$ and $\tau=\mu^{-1}$.

The Additional-Conditioning-Variable Method

Inherent in the method of the previous section are assumptions of existence of derivatives and convergence of series. In particular, it requires that the service-time distribution function $H(x)$ possess a density; this constraint precludes a Maclaurin-series analysis of the important case of constant service times. [However, the results for the constant service-time case follow directly from equation (42).]

In this section we extend the concepts of Burke's additional-conditioning-variable method, whose application to the analysis of the $GI/M/s$ random service queue is summarized above, to the analysis of the $M/G/1$ random-service queue. This method dispenses with the assumptions inherent in the Maclaurin-series method. Thus it is the more general of the two methods. On the other hand, its implementation requires the evaluation, either formal or numerical, of several integrals, so that its relative utility will depend on the particular form of the service-time distribution function $H(x)$ under consideration.

As before, let T_c be the test customer's arrival epoch, and let T_1 be the first service-completion epoch subsequent to T_c. Let S_k be the sum of the first k service times commencing at epoch T_1, and let the additional conditioning variable $\check{X}(x)$ be the value of the largest integer $k(k=0,1,\cdots)$ such that $S_k<x$. We define $\check{W}_{j,k}(t)$ as follows:

$$\check{W}_{j,k}(x)=P\{W(T_1+0)>x|W>0, N(T_1+0)=j, \check{X}(x)=k\},$$
$$(j=2,3,\cdots;k=0,1,\cdots)$$

where, as before, $N(T)$ is the number of customers in the system (including service) at any time T, and $W(T)$ is the remaining waiting time for service to commence for a customer who is waiting at time T.

We have that

$$P\{\check{X}(x)=k\}=H^{*k}(x)-H^{*(k+1)}(x), \qquad (k=0, 1, \cdots; H^{*0}(x)=1; x\geqq 0) \quad (64)$$

where $H^{*k}(x)$ is the k-fold convolution with itself of the service-time distribution function $H(x)$. From the law of total probability it follows that

$$\check{W}_j(x)=\sum_{k=0}^{k=\infty} \check{W}_{j,k}(x) P\{\check{X}(x)=k\}, \quad (65)$$

and hence our problem will be solved if we can determine the conditional probabilities $W_{j,k}(t)$ $(j=2, 3, \cdots; k=0, 1, \cdots)$.

Clearly

$$\check{W}_{j,0}(x)=1. \qquad (j=2, 3, \cdots) \quad (66)$$

Let us define for $k>0$

$$H(\xi|k, x)=P\{S_1\leqq \xi|\check{X}(x)=k\}. \quad (67)$$

It follows from the definition of conditional probability that

$$d_\xi H(\xi|k, x)=P\{\check{X}(x-\xi)=k-1\}dH(\xi)/P\{\check{X}(x)=k\}. \qquad (k=1, 2, \cdots) \quad (68)$$

The following recurrence for $k\geqq 1$ is easily seen to be true:

$$\check{W}_{j,k}(x)=\sum_{i=0}^{i=\infty} [(j+i-2)/(j+i-1)]\int_0^x p_i(\xi)\check{W}_{j+i-1,k-1}(x-\xi)\, d_\xi H(\xi|k, x). \quad (69)$$
$$(j=2, 3, \cdots; k=1, 2, \cdots)$$

The required function $P\{W>t|W>0\}$ can now be determined in principle from (41), complemented by (40), (65), (66), and (69).

As an illustration of the additional-conditioning-variable method, we now calculate $P\{W>t|W>0\}$ for the $M/D/1$ random-service queue. Assume that the service times are of constant duration τ, so that the service-time distribution function is

$$H(x)=\begin{cases}0, & \text{when } x<\tau,\\ 1, & \text{when } x\geqq\tau.\end{cases} \quad (70)$$

Since $H(x)$ does not admit a proper density function, the Maclaurin-expansion method does not apply. With $H(x)$ given by (70), (40) reduces to

$$Q_j'(\xi)=\begin{cases}(1/\tau)\Pi_{j-1}^*, & \text{when } \xi\leqq\tau,\\ 0, & \text{when } \xi>\tau.\end{cases} \qquad (j=1, 2, \cdots) \quad (71)$$

(The stationary state distribution $\{\Pi_j^*\}$ for the $M/D/1$ queue is given in references 11, 12, and 13.)

Then (41) becomes, for $t<\tau$,

$$P\{W>t|W>0\}=1-(t/\tau)+(1/\tau)\sum_{j=2}^{j=\infty} [(j-1)/j]\Pi_{j-1}^*\int_0^t \check{W}_j(t-\xi)\, d\xi, \quad (72)$$
$$(0\leqq t<\tau)$$

and, for $t\geqq\tau$,

$$P\{W>t|W>0\}=(1/\tau)\sum_{j=2}^{j=\infty} [(j-1)/j]\Pi_{j-1}^*\int_0^\tau \check{W}_j(t-\xi)\, d\xi. \qquad (\tau\leqq t<\infty) \quad (73)$$

Since $\check{W}_j(x)$ is a step function, the integrals on the right-hand sides of (72) and (73) can be evaluated, giving

$$P\{W>t|W>0\}=1-(t/\tau)+(t/\tau)\sum_{j=2}^{j=\infty}[(j-1)/j]\Pi_{j-1}^*, \qquad (0\leqq t<\tau) \quad (74)$$

and

$$P\{W>t|W>0\}=\frac{1}{\tau}\sum_{j=2}^{\infty}\frac{j-1}{j}\Pi_{j-1}^*\left\{\left(t-\left[\frac{t}{\tau}\right]\tau\right)\check{W}_j(t) + \left(\tau-t+\left[\frac{t}{\tau}\right]\tau\right)\check{W}_j(t-\tau)\right\}, \qquad (\tau\leqq t<\infty) \quad (75)$$

where $[x]$ is defined as the largest integer not exceeding x. Use of equation (75) requires evaluation of $\check{W}_j(x)(j=2, 3, \cdots)$. Since the service times are of constant duration τ, independent of all other variables, we have

$$H(\xi|k, x)=H(\xi)=\begin{cases}0, & \text{when} \quad \xi<\tau,\\ 1, & \text{when} \quad \xi\geqq\tau,\end{cases} \quad (76)$$

so that (69) becomes

$$\check{W}_{j,k}(x)=\sum_{i=0}^{i=\infty}[(j+i-2)/(j+i-1)]p_i(\tau)\check{W}_{j+i-1,k-1}(x-\tau). \qquad (j=2, 3, \cdots; k=1, 2, \cdots) \quad (77)$$

We have from (64) that

$$P\{\check{X}(x)=k\}=\begin{cases}1, & \text{when} \quad k=[t/\tau],\\ 0, & \text{otherwise}.\end{cases} \quad (78)$$

In light of (78), (65) implies

$$\check{W}_j(x)=\check{W}_{j,[x/\tau]}(x). \qquad (j=2, 3, \cdots) \quad (79)$$

Therefore, if we set $k=[x/\tau]$ in (77) we obtain

$$\check{W}_j(x)=\sum_{i=0}^{i=\infty}[(j+i-2)/(j+i-1)]p_i(\tau)\check{W}_{j+i-1}(x-\tau). \qquad (j=2, 3, \cdots; x\geqq\tau) \quad (80)$$

Equation (80) is complemented by

$$\check{W}_j(x)=1. \qquad (j=2, 3, \cdots; x<\tau) \quad (81)$$

Equations (80) and (81) permit calculation of (75).

Note that (80) and (81) could have been obtained directly from the integral equation (42), as was observed earlier.

It can be shown in a straightforward manner that these results are equivalent to those of Burke,[1] who gives several useful sets of curves for the $M/D/1$ random-service queue.

REFERENCES

1. P. J. Burke, "Equilibrium Delay Distribution for One Channel with Constant Holding Time, Poisson Input, and Random Service," *Bell System Tech. J.* **38,** 1021–1031 (1959).
2. ———, unpublished memorandum (1967).
3. R. B. Cooper, unpublished memorandum (1967).

4. ———, *Congestion Theory and Its Applications to Time Division Telephone Switching Systems*, Dissertation, University of Pennsylvania, 1968.
5. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd ed., Wiley, New York, 1971.
6. J. F. C. Kingman, "On Queues in which Customers are Served in Random Order," *Proc. Cambridge Phil. Soc.* **58,** 79–91 (1962).
7. P. LeGall, *Les Systèmes avec ou sans Attente et les Processus Stochastiques*, Tome 1, Dunod, Paris, 1962.
8. C. Palm, "Väntetider Vid Slumpvis Avverkad Kö," *Tekniska Meddelanden från Kungl. Telegrafstyrelsen*," Specialnummer för Teletrafikteknik, Stockholm, (1946). [Translated in "Research on Telephone Traffic Carried by Full Availability Groups," *Tele*, English Edition, **1,** 1–107 (1957).]
9. F. Pollaczek, "La Loi d'Attente des Appels Téléphoniques," *Comptes Rend.* **222,** 353–355 (1946).
10. J. Riordan, "Delay Curves for Calls Served at Random," *Bell System Tech. J.* **32,** 100–119 (1953).
11. ———, *Stochastic Service Systems*, Wiley, New York, 1962.
12. R. Syski, *Introduction to Congestion Theory in Telephone Systems*, Oliver and Boyd, London, 1960.
13. L. Takács, *Introduction to the Theory of Queues*, Oxford University Press, New York, 1962.
14. ———, "Delay Distributions for Simple Trunk Groups with Recurrent Input and Exponential Service Times," *Bell System Tech. J.* **41,** 311–320 (1962).
15. ———, "Delay Distributions for One Line With Poisson Input, General Holding Times, and Various Orders of Service," *Bell System Tech. J.* **42,** 487–503 (1963).
16. E. Vaulot, "Delais d'Attente des Appels Téléphoniques Traités au Hasard," *Comptes Rend.* **222,** 268–269 (1946).