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An Infinite Sum of Erlang Loss Functions

Problem 93-5*, *by* ROBERT B. COOPER (Florida Atlantic University).

For $x > 0$, let

$$S(x) = \sum_{n=0}^{\infty} B(n, x),$$

where $B(n, x)$ is the Erlang loss function of teletraffic theory [1]:

$$B(n, x) = \frac{x^n/n!}{\sum_{k=0}^{n} x^k/k!}.$$

Prove or disprove the conjecture, based on numerical calculations, that

$$S(x) - \frac{x}{2} \sim a \log x + b,$$

where a and b are constants.

This problem arose in a study of the effect of deletions on the performance of hash-structured files. $S(x)$ equals the expected value of the number of memory locations that must be probed to locate a record known to be in the file ("successful search") when the records (1) reside in the file for a random time of arbitrary distribution before being deleted, (2) arrive for insertion according to a Poisson process, and (3) are inserted into the lowest-numbered unoccupied memory location. (x is the average number of records in the file.)

REFERENCE

[1] A. A. JAGERS AND E. A. VAN DOORN, *Problem* 90-8, SIAM Rev., 33 (1991), pp. 281–282.

Solution by G. F. NEWELL (University of California, Berkeley).

In [1, p. 196] and [2, p. 7] it is shown that, to a first approximation for large x,

$$(1)\qquad B(n, x) \approx \begin{cases} 1 - n/x, & n < x, \\ 0, & n > x, \end{cases}$$

which implies that, to this approximation, $S(x) \approx x/2$ (if we integrate with respect to n) or $x/2 + 1/2$ (if we use a discrete sum).

As a second approximation, for $n - x$ of order $x^{1/2}$, it is shown in [1] that

$$(2)\qquad B(n, x) \approx \phi\left(\frac{n-x}{x^{1/2}}\right) \Big/ x^{1/2} \Phi\left(\frac{n-x}{x^{1/2}}\right)$$

in which $\phi(\cdot)$ and $\Phi(\cdot)$ are the normal probability density and cumulative distribution functions, respectively.

If we presume that this approximation is valid for all n, $n = 0$ to ∞, then one can replace the sum by an integral from $-1/2$ to ∞. The integral of (2) gives

$$S(x) \approx -ln\Phi\left(\frac{-x - 1/2}{x^{1/2}}\right).$$

If we now use the asymptotic form for Φ,

$$\Phi(-y) \sim (2\pi)^{-1/2}(+y)^{-1} \exp(-y^2/2), \qquad -y >> 1$$

then

$$(3)\qquad S(x) \approx x/2 + \tfrac{1}{2} ln(x) + \tfrac{1}{2} ln(2\pi) + \tfrac{1}{2} + 0(1/x),$$

which would seem to confirm Cooper's conjecture. The $ln\, x$ term, however, clearly is coming from the slow convergence of (2) to (1) for $|n-x| >> x^{1/2}$, outside the range where (2) is accurate.

A more accurate approximation for $B(n,x)$ is given in [2].

$$(4) \qquad B(n,x) \approx \begin{cases} n^{1/2}\phi(\kappa)/x\Phi(\kappa) & \text{for } \kappa < 0, \\ (x/n)^n e^{n-x}/(2\pi n)^{1/2}\phi(\kappa) & \text{for } \kappa > 0, \end{cases}$$

with

$$\kappa = (n-x)/n^{1/2}.$$

(There is a slight error in the formula in [2] for $\kappa > 0$.) This is nearly the same as (2) except that the variance of the normal distribution is now n rather than x. This makes little differences over the range $|n-x|$ of order $x^{1/2}$, but it will affect the behavior over the range $0 < n < x - 0(x^{1/2})$, which contributes significantly to the $S(x)$.

For $\kappa > 0$ the $B(n,x)$ decays so rapidly with κ that the approximation (2) is still valid over the range of κ, which contributes appreciably to the $S(x)$ and is even valid over the range $\kappa < 0$, $|k|$ comparable with 1. If we integrate (2) only from some point $x - z$, with z comparable with $x^{1/2}$, to ∞, we obtain a contribution to $S(x)$ of

$$-ln\Phi(-z/x^{1/2}) = z^2/2x + \tfrac{1}{2}\, ln(z^2/x) + \tfrac{1}{2}\, ln(2\pi).$$

Now for the contribution to $S(x)$ from $n = 0$ to $x - z$ we use the asymptotic formula for $\Phi(\kappa)$ in (4) but with the correct form of κ. This gives

$$B(n,x) \approx \left(1 - \frac{n}{x}\right)\left[1 + \frac{n}{x^2(1-n/x)^2} + \cdots\right].$$

The first term of this corresponds to (1) and the sum from $n = 0$ to $x - z$ contributes to $S(x)$ an amount $x/2 + 1/2 - z^2/2x$. The integral of the second term contributes an amount $ln(x/z) - 1$. Thus, all together, we obtain

$$(5) \qquad S(x) \approx x/2 + (\tfrac{1}{2})\, ln\, x - \tfrac{1}{2} + (\tfrac{1}{2})\, ln\, (2\pi).$$

That the value of z cancels out the formula confirms that this procedure is consistent. The $ln\, x$ terms remain unchanged from (3), but the constant term is altered. This indeed confirms Cooper's conjecture.

Also solved by W. B. JORDAN (Scotia, NY).

Editorial comment. Defining $R(x)$ to be $S(x)$ minus Newell's approximation (5), we compute $R(10) = .331099$, $R(100) = .111738$, $R(1000) = .036055$, $R(5000) = .016207$, and $R(10000) = .011474$. [O.G.R.]

REFERENCES

[1] G. F. NEWELL, *Applications of Queueing Theory*, 2nd ed., Chapman and Hall, London, 1982.
[2] ———, *The M/M ∞ Service System with Ranked Services in Heavy Traffic*, Lecture Notes in Economics and Mathematical Systems 231, Springer-Verlag, Berlin, 1984.