Chapter 4, Exercise 1

'Finite-source systems with nonidentical sources.'

a Arguing as in Section 2 of Chapter 3 we find

$$b_2 = \frac{\gamma_2 P(1,0)}{\gamma_2 P(0,0) + \gamma_2 P(1,0)} ,$$

expressing that the probability that source 2 is blocked equals the number of blocked source 2 - calls per unit time divided by the total number of source 2 - calls per unit time. Hence

$$b_2 = \frac{P(1,0)}{P(0,0) + P(1,0)} . \qquad (1)$$

b With source 2 inactive, the system is in fact a one server, one source system, and the sole equilibrium state equation is $\gamma_1 P_0 = \mu_1 P_1$. Given $P_0 + P_1 = 1$, we find $P_1 = (\gamma_1/\mu_1)/(1 + \gamma_1/\mu_1)$. b_2' is defined as the probability that source 2 at a randomly chosen point in time finds the server occupied. Clearly, $b_2' = P_1$. That is, whether blocked customer cleared or delayed (!),

$$b_2' = \frac{\gamma_1/\mu_1}{1 + (\gamma_1/\mu_1)} . \qquad (2)$$

c Blocked customers cleared.

First we calculate source 2's blocking probability b_2. The conservation-of-flow equations when both sources are active are

$$(\gamma_1 + \gamma_2) P(0,0) = \mu_1 P(1,0) + \mu_2 P(0,1)$$
$$\mu_1 P(1,0) = \gamma_1 P(0,0)$$
$$\mu_2 P(0,1) = \gamma_2 P(0,0)$$

We need only $P(1,0)$ in terms of $P(0,0)$. The middle equation gives us $P(1,0) = (\gamma_1/\mu_1) P(0,0)$. By (1) then

$$b_2 = \frac{(\gamma_1/\mu_1) P(0,0)}{P(0,0) + (\gamma_1/\mu_1) P(0,0)} = \frac{\gamma_1/\mu_1}{1 + (\gamma_1/\mu_1)} . \qquad (3)$$

A comparison with Eq. (2) shows that in the BCC case $b_2 = b_2'$.

(Chap. 4, Ex. 1)

$\boxed{d}$ Blocked customers delayed.

Again we calculate source 2's blocking probability b_2. The conservation-of-flow equations are those found in Section 4.1 above. Omitting one equation we have

$$(\gamma_2+\mu_1)P(1,0) = \gamma_1 P(0,0) + \mu_2 P(2,1),$$

$$(\gamma_1+\mu_2)P(0,1) = \gamma_2 P(0,0) + \mu_1 P(1,2),$$

$$\mu_1 P(1,2) = \gamma_2 P(1,0),$$

$$\mu_2 P(2,1) = \gamma_1 P(0,1).$$

Substituting the last two equations into the first two, and then eliminating $P(0,1)$ and solving for $P(1,0)$ we derive

$$P(1,0) = \frac{\gamma_1(\mu_2+\gamma_1+\gamma_2)}{\mu_1\mu_2+\mu_1\gamma_1+\mu_2\gamma_2} P(0,0).$$

Substitution of this expression into Eq. (1) gives

$$b_2 = \frac{\gamma_1[\mu_2+\gamma_1+\gamma_2]}{\mu_1\mu_2+\mu_1\gamma_1+\mu_2\gamma_2+\gamma_1[\mu_2+\gamma_1+\gamma_2]},$$

or,

$$b_2 = \frac{(\gamma_1/\mu_1)[\mu_2+\gamma_1+\gamma_2]}{[\mu_2+\gamma_1+(\mu_2/\mu_1)\gamma_2]+(\gamma_1/\mu_1)[\mu_2+\gamma_1+\gamma_2]}. \qquad (4)$$

We shall prove that $b_2 = b_2'$ if and only if $\mu_1=\mu_2$. First assume $\mu_1=\mu_2$. Then Eq. (4) reduces to $b_2 = (\gamma_1/\mu_1)/(1+\gamma_1/\mu_1)$ $= b_2'$, by Eq. (2). Conversely, assume $b_2=b_2'$. By Equations (2) and (4) this implies

$$\frac{\mu_2+\gamma_1+\gamma_2}{\mu_1\mu_2+\mu_1\gamma_1+\mu_2\gamma_2+\gamma_1\mu_2+\gamma_1^2+\gamma_1\gamma_2} = \frac{1}{\mu_1+\gamma_1} \qquad [b_2=b_2']$$

it follows easily that $\mu_1=\mu_2$. We conclude that in this particular BCD model with nonidentical sources, the arriving customer's 2-source distribution and his observer's 1-source distribution are the same if and only if $\mu_1=\mu_2$. $\square$

Chapter 4, Exercise 2

'a. Three cities A, B, and C, are interconnected by two trunk groups;'

In every case, let λ_i, μ_i, j_i denote arrival rate, completion rate, and number of calls in progress, respectively, for city connection no. i $(i=1,2,3)$, where $i=1$ refers to A−B, $i=2$ refers to B−C, $i=3$ refers to A−C. Let $P(j_1,j_2,j_3)$ be the equilibrium state probability of state (j_1,j_2,j_3). Always, it is understood that $j_1, j_2, j_3 \geq 0$.

$\boxed{a}$ In case (a) the equilibrium state equations are:

$$(\lambda_1+\lambda_2+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)\,P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)+\lambda_3 P(j_1,j_2,j_3-1) + (j_1+1)\mu_1 P(j_1+1,j_2,j_3)+(j_2+1)\mu_2 P(j_1,j_2+1,j_3)+(j_3+1)\mu_3 P(j_1,j_2,j_3+1) \quad \begin{pmatrix} j_1+j_3<s_1,\\ j_2+j_3<s_2 \end{pmatrix}$$

$$(\lambda_1+j_1\mu_1+j_2\mu_2+j_3\mu_3)\,P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)+\lambda_3 P(j_1,j_2,j_3-1) + (j_1+1)\mu_1 P(j_1+1,j_2,j_3) \quad \begin{pmatrix} j_1+j_3<s_1\\ j_2+j_3=s_2 \end{pmatrix}$$

$$(\lambda_2+j_1\mu_1+j_2\mu_2+j_3\mu_3)\,P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)+\lambda_3 P(j_1,j_2,j_3-1) + (j_2+1)\mu_2 P(j_1,j_2+1,j_3) \quad \begin{pmatrix} j_1+j_3=s_1\\ j_2+j_3<s_2 \end{pmatrix}$$

$$(j_1\mu_1+j_2\mu_2+j_3\mu_3)\,P(j_1,j_2,j_3) = \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)+\lambda_3 P(j_1,j_2,j_3-1) \quad \begin{pmatrix} j_1+j_3=s_1\\ j_2+j_3=s_2 \end{pmatrix}$$

If $s_1=\infty$ and $s_2=\infty$, then j_1, j_2 and j_3 are independent Poisson variables, and

$$P(j_1,j_2,j_3) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\,\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\,\frac{(\lambda_3/\mu_3)^{j_3}}{j_3!}\cdot c \qquad [s_1=\infty,\ s_2=\infty].$$

Because of a correspondence between terms on left- and right-hand sides of all the equilibrium state equations in case (a) it is clear that also in the present case the solution has the form

$$P(j_1,j_2,j_3) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\,\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\,\frac{(\lambda_3/\mu_3)^{j_3}}{j_3!}\cdot c \quad \begin{pmatrix} 0\leq j_1+j_3\leq s_1,\\ 0\leq j_2+j_3\leq s_2 \end{pmatrix}.$$

c is found from the normalization equation $\sum P(j_1,j_2,j_3)=1$.

(Chap. 4, Ex. 2 b)

$\boxed{b}$ In case (b) the equilibrium state equations are:

$$\begin{aligned}(\lambda_1+\lambda_2+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) &= \\ \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)& \\ +\lambda_3 P(j_1,j_2,j_3-1)+(j_1+1)\mu_1 P(j_1+1,j_2,j_3)& \\ +(j_2+1)\mu_2 P(j_1,j_2+1,j_3)+(j_3+1)\mu_3 P(j_1,j_2,j_3+1)&\end{aligned}\qquad \begin{pmatrix} j_1+\max(0,j_3-s)<s_1, \\ j_2+\max(0,j_3-s)<s_2 \end{pmatrix}$$

$$\begin{aligned}(\lambda_1+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) &= \\ \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)& \\ +\lambda_3 P(j_1,j_2,j_3-1)+(j_1+1)\mu_1 P(j_1+1,j_2,j_3)& \\ +(j_3+1)\mu_3 P(j_1,j_2,j_3+1)&\end{aligned}\qquad (j_1<s_1,\ j_2=s_2,\ j_3<s)$$

$$\begin{aligned}(\lambda_2+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) &= \\ \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)& \\ +\lambda_3 P(j_1,j_2,j_3-1)+(j_2+1)\mu_2 P(j_1,j_2+1,j_3)& \\ +(j_3+1)\mu_3 P(j_1,j_2,j_3+1)&\end{aligned}\qquad (j_1=s_1,\ j_2<s_2,\ j_3<s)$$

$$\begin{aligned}(\lambda_1+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) &= \\ \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)& \\ +\lambda_3 P(j_1,j_2,j_3-1)+(j_1+1)\mu_1 P(j_1+1,j_2,j_3)&\end{aligned}\qquad \begin{pmatrix} j_1+(j_3-s)<s_1, \\ j_2+(j_3-s)=s_2, \\ j_3\geq s \end{pmatrix}$$

$$\begin{aligned}(\lambda_2+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) &= \\ \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)& \\ +\lambda_3 P(j_1,j_2,j_3-1)+(j_2+1)\mu_2 P(j_1,j_2+1,j_3)&\end{aligned}\qquad \begin{pmatrix} j_1+(j_3-s)=s_1, \\ j_2+(j_3-s)<s_2, \\ j_3\geq s \end{pmatrix}$$

$$\begin{aligned}(\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) &= \\ \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)& \\ +\lambda_3 P(j_1,j_2,j_3-1)+(j_3+1)\mu_3 P(j_1,j_2,j_3+1)&\end{aligned}\qquad (j_1=s_1,\ j_2=s_2,\ j_3<s)$$

$$\begin{aligned}(j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) &= \\ \lambda_1 P(j_1-1,j_2,j_3)+\lambda_2 P(j_1,j_2-1,j_3)& \\ +\lambda_3 P(j_1,j_2,j_3-1).&\end{aligned}\qquad \begin{pmatrix} j_1+(j_3-s)=s_1, \\ j_2+(j_3-s)=s_2 \\ j_3\geq s \end{pmatrix}$$

The solution is of the same type as in case (a), namely

$$P(j_1,j_2,j_3) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\times\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\times\frac{(\lambda_3/\mu_3)^{j_3}}{j_3!}c \quad \begin{pmatrix} 0\leq j_1+\max(0,j_3-s)\leq s_1, \\ 0\leq j_2+\max(0,j_3-s)\leq s_2 \end{pmatrix}$$

(Chap. 4, Ex. 2 c)

$\boxed{c}$ Without a switching capability the state description must be (j_1, j_2, j_3', j_3'') where j_3' and j_3'' denote directly and indirectly connected calls between A and C. This complicates the equilibrium state equations somewhat, but worse, the decomposition property is lost, so that the solution method above is not applicable

$\boxed{d}$ In case d, denote by s_3 the number of trunks directly connecting A and C. Observe that $j_1 > s_1 \Rightarrow \{j_2 < s_2, j_3 < s_3\}$ with similar implications of $j_2 > s_2$ and $j_3 > s_3$. The equilibrium state equations are:

$$\begin{aligned}(\lambda_1+\lambda_2+\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = {}& \\ \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3)& \\ + \lambda_3 P(j_1,j_2,j_3-1) + (j_1+1)\mu_1 P(j_1+1,j_2,j_3)& \\ + (j_2+1)\mu_2 P(j_1,j_2+1,j_3) + (j_3+1)\mu_3 P(j_1,j_2,j_3+1)&\end{aligned} \qquad \begin{aligned}&\begin{pmatrix} j_1+\max(0,j_3-s_3)<s_1, \\ j_2+\max(0,j_3-s_3)<s_2\end{pmatrix}, \\ \text{or } &\begin{pmatrix} j_1+\max(0,j_2-s_2)<s_1, \\ j_3+\max(0,j_2-s_2)<s_3\end{pmatrix}, \\ \text{or } &\begin{pmatrix} j_2+\max(0,j_1-s_1)<s_2, \\ j_3+\max(0,j_1-s_1)<s_3\end{pmatrix}.\end{aligned}$$

$$\begin{aligned}(\lambda_1+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = {}& \\ \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3)& \\ + \lambda_3 P(j_1,j_2,j_3-1) + (j_1+1)\mu_1 P(j_1+1,j_2,j_3)&\end{aligned} \qquad \begin{aligned}&(j_1+(j_3-s_3)<s_1,\ j_2+(j_3-s_3)=s_2,\ j_3\geqq s_3), \\ \text{or } &(j_1+(j_2-s_2)<s_1,\ j_2\geqq s_2,\ j_3+(j_2-s_2)=s_3).\end{aligned}$$

$$\begin{aligned}(\lambda_2+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = {}& \\ \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3)& \\ + \lambda_3 P(j_1,j_2,j_3-1) + (j_2+1)\mu_2 P(j_1,j_2+1,j_3)&\end{aligned} \qquad \begin{aligned}&(j_1+(j_3-s_3)=s_1,\ j_2+(j_3-s_3)<s_2,\ j_3\geqq s_3), \\ \text{or } &(j_1\geqq s_1,\ j_2+(j_1-s_1)<s_2,\ j_3+(j_1-s_1)=s_3).\end{aligned}$$

$$\begin{aligned}(\lambda_3+j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = {}& \\ \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3)& \\ + \lambda_3 P(j_1,j_2,j_3-1) + (j_3+1)\mu_3 P(j_1,j_2,j_3+1)&\end{aligned} \qquad \begin{aligned}&(j_1+(j_2-s_2)=s_1,\ j_2\geqq s_2,\ j_3+(j_2-s_2)<s_3), \\ \text{or } &(j_1\geqq s_1,\ j_2+(j_1-s_1)=s_2,\ j_3+(j_1-s_1)<s_3).\end{aligned}$$

$$\begin{aligned}(j_1\mu_1+j_2\mu_2+j_3\mu_3)P(j_1,j_2,j_3) = {}& \\ \lambda_1 P(j_1-1,j_2,j_3) + \lambda_2 P(j_1,j_2-1,j_3)& \\ + \lambda_3 P(j_1,j_2,j_3-1)&\end{aligned} \qquad \begin{aligned}&(j_1\geqq s_1,\ j_2+(j_1-s_1)=s_2,\ j_3+(j_1-s_1)=s_3), \\ \text{or } &(j_1+(j_2-s_2)=s_1,\ j_2\geqq s_2,\ j_3+(j_2-s_2)=s_3), \\ \text{or } &(j_1+(j_3-s_3)=s_1,\ j_2+(j_3-s_3)=s_2,\ j_3\geqq s_3).\end{aligned}$$

Again, the solution is of the same type as in case (a), namely

$$P(j_1,j_2,j_3) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\frac{(\lambda_3/\mu_3)^{j_3}}{j_3!}c$$

for all feasible combinations (j_1,j_2,j_3). As usual, c is found from the condition $\sum P(j_1,j_2,j_3) = 1$. $\square$

Chapter 4, Exercise 3

'A group of s trunks serves two types of Poisson traffic on a BCC..'

Let j_1 be the number of ordinary calls and j_2 the number of wideband calls in progress. Then j_1+kj_2 is the total number of trunks that are held. The equilibrium state equations are as follows, assuming $j_1 \geq 0$ and $j_2 \geq 0$,

$$(\lambda_1+\lambda_2+j_1\mu_1+j_2\mu_2)P(j_1,j_2) = \lambda_1 P(j_1-1,j_2)+\lambda_2 P(j_1,j_2-1) + (j_1+1)\mu_1 P(j_1+1,j_2)+(j_2+1)\mu_2 P(j_1,j_2+1) \qquad (0 \leq j_1+kj_2 \leq s-k)$$

$$(\lambda_1+j_1\mu_1+j_2\mu_2)P(j_1,j_2) = \lambda_1 P(j_1-1,j_2)+\lambda_2 P(j_1,j_2-1) + (j_1+1)\mu_1 P(j_1+1,j_2) \qquad (s-k < j_1+kj_2 < s)$$

$$(j_1\mu_1+j_2\mu_2)P(j_1,j_2) = \lambda_1 P(j_1-1,j_2)+\lambda_2 P(j_1,j_2-1). \qquad (j_1+kj_2 = s)$$

By the same considerations as before, it is seen that the solution is given by

$$P(j_1,j_2) = \frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}C \qquad (0 \leq j_1+kj_2 \leq s),$$

where

$$C = \left[\sum_{0\leq j_1+kj_2\leq s}\frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\right]^{-1}.$$

Denote by $j_2^* = [\frac{s}{k}]$ the highest possible number of wideband calls in progress. Let P_1 be the probability that an ordinary call is lost, and let P_2 be the probability that a wideband call is lost. Then, clearly,

$$P_1 = P\{j_1+kj_2 = s\} = \sum_{j_2=0}^{j_2^*} P(s-kj_2,j_2),$$

and

$$P_2 = P\{s-k+1 \leq j_1+kj_2 \leq s\} = \sum_{j_2=0}^{j_2^*-1}\sum_{j_1=s-j_2k-(k-1)}^{s-j_2k} P(j_1,j_2) + \sum_{j_1=0}^{s-j_2^*k} P(j_1,j_2^*). \qquad \square$$

Chapter 4, Exercise 4

'A group of s servers handles n types of customers.'

Let $I(x)=0$ if $x \leq 0$, $I(x)=1$ if $x>0$. With this notation the equilibrium state equations can be written

$$\left(\sum_{i=1}^{n}\lambda_i I(k_i-j_i)+\sum_{i=1}^{n} j_i\mu_i\right)P(j_1,j_2,\dots,j_n)= \qquad \begin{pmatrix} 0\leq j_i\leq k_i,\ i=1,\dots,n; \\ 0\leq \Sigma_{i=1}^{n} j_i<s \end{pmatrix}$$
$$\sum_{i=1}^{n}\lambda_i P(\dots,j_{i-1},j_i-1,j_{i+1},\dots)$$
$$+\sum_{i=1}^{n}(j_i+1)\mu_i I(k_i-j_i)P(\dots,j_{i-1},j_i+1,j_{i+1},\dots),$$

$$\left(\sum_{i=1}^{n} j_i\mu_i\right)P(j_1,j_2,\dots,j_n)=\sum_{i=1}^{n}\lambda_i P(\dots,j_{i-1},j_i-1,j_{i+1},\dots) \qquad \begin{pmatrix} 0\leq j_i\leq k_i, \\ i=1,\dots,n; \\ \Sigma_{i=1}^{n} j_i=s \end{pmatrix},$$

where j_i denotes the number of customers of type i, and $j_i \geq 0$.

The correspondence between LHS and RHS terms such as $\lambda_i I(k_i-j_i)P(j_1,j_2,\dots,j_n)$ and $(j_i+1)\mu_i I(k_i-j_i)P(\dots,j_{i-1},j_i+1,j_{i+1})$ once more indicates a solution of the form

$$P(j_1,j_2,\dots,j_n)=\frac{(\lambda_1/\mu_1)^{j_1}}{j_1!}\frac{(\lambda_2/\mu_2)^{j_2}}{j_2!}\cdots\frac{(\lambda_n/\mu_n)^{j_n}}{j_n!}\times c \qquad \begin{pmatrix} 0\leq j_i\leq k_i,\ i=1,\dots,n; \\ 0\leq \Sigma_{i=1}^{n} j_i\leq s \end{pmatrix}$$

where, as usual, c is determined from $\Sigma\, P(j_1,j_2,\dots,j_n)$.

Let P_0 be the equilibrium probability that all servers are busy, and let P_i be the probability that $j_i=k_i$ while not all the servers are busy. Obviously,

$$P_0=\sum_{(j_1,\dots,j_n)\in S_0} P(j_1,j_2,\dots,j_n), \qquad (1)$$

where

$$S_0=\{(j_1,j_2,\dots,j_n): 0\leq j_i\leq k_i,\ i=1,\dots,n; \sum_{i=1}^{n} j_i=s\},$$

and

$$P_i=\sum_{(j_1,\dots,j_n)\in S_i} P(j_1,j_2,\dots,j_n), \qquad (2)$$

where

$$S_i=\{(j_1,j_2,\dots,j_n): j_i=k_i;\ 0\leq j_r\leq k_r,\ r\neq i;\ \sum_{i=1}^{n} j_i<s\}.$$

By the assumption of Poisson arrival streams, the probability $\hat{P}_i$ that a customer of type i will be blocked equals

$$\hat{P}_i=P_0+P_i. \qquad (3)$$

□

Chapter 4, Exercise 5

'A group of s servers handles two types of customers on a BCC basis.'

$\boxed{a}$ The equilibrium state equations are, for $j_1, j_2 \geq 0$,

$$(\lambda + (n-j_2)\gamma + j_1\mu_1 + j_2\mu_2) P_n(j_1, j_2) = \lambda P_n(j_1-1, j_2) + (n-j_2+1)\gamma P_n(j_1, j_2-1) + (j_1+1)\mu_1 P_n(j_1+1, j_2) + (j_2+1)\mu_2 P_n(j_1, j_2+1) \qquad \begin{pmatrix} 0 \leq j_1+j_2 < s, \\ j_2 < \min(s,n) \end{pmatrix}$$

$$(\lambda + j_1\mu_1 + j_2\mu_2) P_n(j_1, j_2) = \lambda P_n(j_1-1, j_2) + (n-j_2+1)\gamma P_n(j_1, j_2-1) + (j_1+1)\mu_1 P_n(j_1+1, j_2) \qquad \begin{pmatrix} 0 \leq j_1+j_2 < s, \\ j_2 = \min(s,n) \end{pmatrix}$$

$$(j_1\mu_1 + j_2\mu_2) P_n(j_1, j_2) = \lambda P_n(j_1-1, j_2) + (n-j_2+1)\gamma P_n(j_1, j_2-1). \qquad \begin{pmatrix} j_1+j_2 = s, \\ j_2 \leq \min(s,n) \end{pmatrix}$$

In case $s = \infty$, the equilibrium states j_1 and j_2 are independent, $\lambda P^{(1)}(j_1) = (j_1+1)\mu_1 P^{(1)}(j_1+1)$ and $(n-j_2)\gamma P^{(2)}(j_2) = (j_2+1)\mu_2 P^{(2)}(j_2+1)$, by which $P^{(1)}(j_1) = [(\lambda/\mu_1)^{j_1}/j_1!]c^{(1)}$ and $P^{(2)}(j_2) = \binom{n}{j_2}(\gamma/\mu_2)^{j_2}c^{(2)}$, $j_2 \leq n$. This suggests that in the present case where $s < \infty$ we will have the product solution

$$P_n(j_1, j_2) = \frac{(\lambda/\mu_1)^{j_1}}{j_1!}\binom{n}{j_2}\left(\frac{\gamma}{\mu_2}\right)^{j_2} c_n \qquad \begin{pmatrix} 0 \leq j_1+j_2 \leq s, \\ j_2 \leq \min(s,n) \end{pmatrix} \qquad (1)$$

for all feasible (j_1, j_2), where c_n is determined from the condition $\sum_{j_2=0}^{\min(s,n)} \sum_{j_1=0}^{s-j_2} P_n(j_1, j_2) = 1$. Equation (1) is verified by noting that the equilibrium state equations can be decomposed into equations of the two types $\lambda P_n(j_1, j_2) = (j_1+1)\mu_1 P_n(j_1+1, j_2)$ and $(n-j_2)\gamma P_n(j_1, j_2) = (j_2+1)\mu_2 P_n(j_1, j_2+1)$ which are both satisfied by (1).

$\boxed{b}$ Customers of type 1 arrive in a Poisson stream. Therefore $\Pi'_n(j_1, j_2; t) = P_n(j_1, j_2; t)$ and

$$\Pi'_n(j_1, j_2) = P_n(j_1, j_2) \qquad \begin{pmatrix} 0 \leq j_1+j_2 \leq s, \\ j_2 \leq \min(s,n) \end{pmatrix} \qquad (2)$$

By an analogy to Eq. (2.6) of Chapter 3 the equilibrium probability that a customer of type 2 will arrive in state (j_1, j_2) is

(Chap. 4, Ex. 5 b)

$$\Pi_n^2(j_1,j_2) = \frac{(n-j_2)\gamma P_n(j_1,j_2)}{\sum_{k_2=0}^{\min(s,n)} \sum_{k_1=0}^{s-k_2} (n-k_2)\gamma P_n(k_1,k_2)} \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s, \\ j_2 \le \min(s,n) \end{pmatrix}$$

By (1), then,

$$\Pi_n^2(j_1,j_2) = \frac{(n-j_2)\frac{(\lambda/\mu_1)^{j_1}}{j_1!}\binom{n}{j_2}\left(\frac{\gamma}{\mu_2}\right)^{j_2}}{\sum_{k_2=0}^{\min(s,n)} \sum_{k_1=0}^{s-k_2} (n-k_2)\frac{(\lambda/\mu_1)^{k_1}}{k_1!}\binom{n}{k_2}\left(\frac{\gamma}{\mu_2}\right)^{k_2}} \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s, \\ j_2 \le \min(s,n) \end{pmatrix} \quad (*)$$

We shall deal with the two cases $n \le s$ and $n > s$ in turn.

$\underline{n \le s.}$ Here $\min(s,n) = n$. Obviously, $\Pi_n^2(j_1,n) = P_{n-1}(j_1,n) = 0$. For $j_2 \le n-1 = \min(s,n-1)$, Eq. (*) gives, after rewriting,

$$\Pi_n^2(j_1,j_2) = \frac{\frac{(\lambda/\mu_1)^{j_1}}{j_1!}\binom{n-1}{j_2}\left(\frac{\gamma}{\mu_2}\right)^{j_2}}{\sum_{k_2=0}^{\min(s,n-1)} \sum_{k_1=0}^{s-k_2} \frac{(\lambda/\mu_1)^{k_1}}{k_1!}\binom{n-1}{k_2}\left(\frac{\gamma}{\mu_2}\right)^{k_2}} \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s, \\ j_2 \le \min(s,n-1) \end{pmatrix}$$

Comparison with Eq. (1) shows that $\Pi_n^2(j_1,j_2) = P_{n-1}(j_1,j_2)$ for $0 \le j_1+j_2 \le s$ and $j_2 \le \min(s,n-1)$. Thus we have shown that if $n \le s$, then $\Pi_n^2(j_1,j_2) = P_{n-1}(j_1,j_2)$ for $0 \le j_1+j_2 \le s$ and $j_2 \le \min(s,n)$.

$\underline{n > s.}$ Here $\min(s,n) = \min(s,n-1)$ $(=s)$ and $n-j_2 > 0$ for all feasible j_2. By (*) it follows that, again,

$$\Pi_n^2(j_1,j_2) = \frac{\frac{(\lambda/\mu_1)^{j_1}}{j_1!}\binom{n-1}{j_2}\left(\frac{\gamma}{\mu_2}\right)^{j_2}}{\sum_{k_2=0}^{\min(s,n-1)} \sum_{k_1=0}^{s-k_2} \frac{(\lambda/\mu_1)^{k_1}}{k_1!}\binom{n-1}{k_2}\left(\frac{\gamma}{\mu_2}\right)^{k_2}} \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s \\ j_2 \le \min(s,n-1) \end{pmatrix}$$

Comparison with Eq. (1) shows that also if $n > s$, then $\Pi_n^2(j_1,j_2) = P_{n-1}(j_1,j_2)$ for $0 \le j_1+j_2 \le s$ and $j_2 \le \min(s,n)$. Note that, despite appearances, the range of j_2 is not the same for $\Pi_n^2(j_1,j_2)$ when $n \le s$ and when $n > s$, since in the latter case we have just made the substitution $\min(s,n) = \min(s,n-1)$.

We conclude that, for any $n \ge 1$,

$$\Pi_n^2(j_1,j_2) = P_{n-1}(j_1,j_2) \qquad \begin{pmatrix} 0 \le j_1+j_2 \le s, \\ j_2 \le \min(s,n) \end{pmatrix}. \qquad (3)$$

$\square$

Chapter 4, Exercise 6

'Calls arrive according to a Poisson process...'

For $j_1, j_2 \geq 0$, the equilibrium state equations are

$$\left(\frac{n_1-j_1}{n_1+n_2-j_1-j_2}\lambda + \frac{n_2-j_2}{n_1+n_2-j_1-j_2}\lambda + j_1\mu + j_2\mu\right) P(j_1,j_2) = \qquad (j_1<s_1,\ j_2<s_2)$$
$$\frac{n_1-j_1+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1-1,j_2) + \frac{n_2-j_2+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1,j_2-1)$$
$$+ (j_1+1)\mu P(j_1+1,j_2) + (j_2+1)\mu P(j_1,j_2+1)$$

$$\left(\frac{n_2-j_2}{n_1+n_2-j_1-j_2}\lambda + j_1\mu + j_2\mu\right) P(j_1,j_2) = \qquad (j_1=s_1,\ j_2<s_2)$$
$$\frac{n_1-j_1+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1-1,j_2) + \frac{n_2-j_2+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1,j_2-1)$$
$$+ (j_2+1)\mu P(j_1,j_2+1)$$

$$\left(\frac{n_1-j_1}{n_1+n_2-j_1-j_2}\lambda + j_1\mu + j_2\mu\right) P(j_1,j_2) = \qquad (j_1<s_1,\ j_2=s_2)$$
$$\frac{n_1-j_1+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1-1,j_2) + \frac{n_2-j_2+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1,j_2-1)$$
$$+ (j_1+1)\mu P(j_1+1,j_2)$$

$$\left(j_1\mu + j_2\mu\right) P(j_1,j_2) = \qquad (j_1=s_1,\ j_2=s_2)$$
$$\frac{n_1-j_1+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1-1,j_2) + \frac{n_2-j_2+1}{n_1+n_2-j_1-j_2+1}\lambda P(j_1,j_2-1).$$

Clearly, a solution to the equations

$$\frac{n_1-j_1}{n_1+n_2-j_1-j_2}\lambda P(j_1,j_2) = (j_1+1)\mu P(j_1+1,j_2) \qquad \begin{pmatrix} 0 \leq j_1 < s_1, \\ 0 \leq j_2 \leq s_2 \end{pmatrix}, \qquad (*)$$

$$\frac{n_2-j_2}{n_1+n_2-j_1-j_2}\lambda P(j_1,j_2) = (j_2+1)\mu P(j_1,j_2+1) \qquad \begin{pmatrix} 0 \leq j_1 \leq s_1, \\ 0 \leq j_2 < s_2 \end{pmatrix}, \qquad (**)$$

will also be a solution to the equilibrium state equations above.

(Chap. 4, Ex. 6)

By recursive solution, (*) yields

$$P(j_1,0) = \frac{n_1!(n_1+n_2-j_1)!}{j_1!\,(n_1+n_2)!\,(n_1-j_1)!}\left(\frac{\lambda}{\mu}\right)^{j_1} P(0,0) \qquad (j_1=0,1,\dots,s_1),$$

and (**) yields

$$P(j_1,j_2) = \frac{n_2!(n_1+n_2-j_1-j_2)!}{j_2!(n_1+n_2-j_1)!\,(n_2-j_2)!}\left(\frac{\lambda}{\mu}\right)^{j_2} P(j_1,0) \qquad \begin{pmatrix} j_1=0,1,\dots,s_1, \\ j_2=0,1,\dots,s_2 \end{pmatrix}.$$

Combination of the two equations and the substitution $\frac{\lambda}{\mu}=a$ result in

$$P(j_1,j_2) = \frac{\binom{n_1}{j_1}\binom{n_2}{j_2}}{\binom{n_1+n_2}{j_1+j_2}} \times \frac{a^{j_1+j_2}}{(j_1+j_2)!} P(0,0) \qquad \begin{pmatrix} 0\leq j_1 \leq s_1 \\ 0 \leq j_2 \leq s_2 \end{pmatrix}.$$

When $(n_1,n_2)=(s_1,s_2)$ no call is lost if any trunk is idle, and every call is lost if all trunks are occupied. By (3.3) of Chapter 3 then $P_{j_1+j_2} = [a^{j_1+j_2}/(j_1+j_2)!]\,P_0$. Furthermore, in this case, an arriving customer will with equal probability be directed to every idle trunk. Consequently, the j_1+j_2 occupied trunks are actually drawn at random from the n_1+n_2 trunks. By the hypergeometric distribution $P(j_1|j_1+j_2) = \binom{n_1}{j_1}\binom{n_2}{j_2}/\binom{n_1+n_2}{j_1+j_2}$. It follows that, for $(n_1,n_2)=(s_1,s_2)$, $P(j_1,j_2) = P(j_1|j_1+j_2)\cdot P_{j_1+j_2}$ in agreement with the derived formula for $P(j_1,j_2)$. This may suggest that the formula holds also for $n_1 \geq s_1$ and $n_2 \geq s_2$, as it does.

Chapter 4, Exercise 7

'Customers arrive according to a Poisson process...'

$\boxed{a}$ By the theorem of total probability,

$$\sum_{x_1+\dots+x_s=j} \tilde{P}(x_1,\dots,x_s) = P_j \qquad (j=0,1,\dots,s)$$

for any $\{\mu_i\}$. If $\mu_1=\dots=\mu_s=\mu$, then the model specializes to the Erlang loss model, with P_j given by Eq. (3.3) of Chapter 3.

(Chap. 4, Ex. 7a)

With random server selection and $\mu_1 = \cdots = \mu_s = \mu$, clearly all combinations of j busy servers have equal probability. Since the number of combinations is $\binom{s}{j}$,

$$\tilde{P}(x_1,\ldots,x_s) = \binom{s}{j}^{-1} P_j \qquad (x_1+\ldots+x_s = j;\ \mu_1=\cdots=\mu_s=\mu).$$

$\boxed{b}$ For $x_i \in \{0,1\}$, the equilibrium state equations may be written:

$$\Big(\lambda \tfrac{1}{s-j} \sum_{i=1}^{s}(1-x_i) + \sum_{i=1}^{s} x_i\mu_i\Big)\,\tilde{P}(x_1,\ldots,x_s) = \qquad \Big(0 \le \sum_{i=1}^{s} x_i = j < s\Big)$$
$$\lambda \tfrac{1}{s-j+1}\Big(\tilde{P}(x_1-1,x_2,\ldots,x_s) + \tilde{P}(x_1,x_2-1,\ldots,x_s) + \cdots + \tilde{P}(x_1,\ldots,x_{s-1},x_s-1)\Big)$$
$$+ (x_1+1)\mu_1\tilde{P}(x_1+1,x_2,\ldots,x_s) + \cdots + (x_i+1)\mu_i\tilde{P}(x_1,\ldots,x_i+1,\ldots,x_s) + \cdots + (x_s+1)\mu_s\tilde{P}(x_1,\ldots,x_{s-1},x_s+1),$$

$$\Big(\sum_{i=1}^{s} x_i\mu_i\Big)\,\tilde{P}(x_1,\ldots,x_s) = \qquad \Big(\sum_{i=1}^{s} x_i = s\Big)$$
$$\lambda\Big(\tilde{P}(x_1-1,x_2,\ldots,x_s) + \tilde{P}(x_1,x_2-1,\ldots,x_s) + \cdots + \tilde{P}(x_1,\ldots,x_{s-1},x_s-1)\Big)$$

Once more, there is a pairwise correspondence between LHS and RHS terms. It is seen that the equilibrium state equations will be satisfied by $\tilde{P}(x_1,\ldots,x_s)$ satisfying

$$\lambda \tfrac{1}{s-j}\,\tilde{P}(x_1,\ldots,x_k,\ldots,x_s) = (x_k+1)\mu_k\tilde{P}(x_1,\ldots,x_k+1,\ldots,x_s) \quad \Big(x_k=0,\ 0\le\sum_{i=1}^{s}x_i=j<s\Big) \quad (*)$$

Let $\tilde{P}_0 = \tilde{P}(0,0,\ldots,0)$. By $(*)$, $\tilde{P}(x_1,\ldots,x_k,\ldots,x_s) = \frac{1}{s}\frac{\lambda}{\mu_k}\tilde{P}_0$ for $x_k=1$ and $\sum_{i=1}^{s} x_i = j = 1$ (ie. $x_i=0$ for $i\ne k$). This finding can be expressed:

$$\tilde{P}(x_1,\ldots,x_s) = \frac{(s-1)!}{s!}\prod_{i=1}^{s}\Big(\frac{\lambda}{\mu_i}\Big)^{x_i}\tilde{P}_0 \qquad \Big(\sum_{i=1}^{s}x_i = j = 1,\ x_i\in\{0,1\}\Big).$$

By recursion, $(*)$ yields

$$\tilde{P}(x_1,\ldots,x_s) = \frac{(s-j)!}{s!}\prod_{i=1}^{s}\Big(\frac{\lambda}{\mu_i}\Big)^{x_i}\tilde{P}_0 \qquad \Big(\sum_{i=1}^{s}x_i = j\ (j=1,2,\ldots,s),\ x_i\in\{0,1\}\Big)$$

Notice, the formula also holds for $\sum_{i=1}^{s} x_i = j = 0$. Finally, rewriting this formula we obtain

$$\tilde{P}(x_1,\ldots,x_s) = \binom{s}{\Sigma x_i}^{-1} \frac{\prod_{i=1}^{s}\big(\frac{\lambda}{\mu_i}\big)^{x_i}}{(\Sigma x_i)!}\,\tilde{P}_0 \qquad (x_i\in\{0,1\},\ i=1,\ldots,s)$$

As usual, $\tilde{P}_0$ is found by the normalization condition. □

Chapter 4, Exercise 8

'Network of queues.'

$\boxed{a}$ Let λ_i denote the mean arrival rate at Q_i. Obviously,

$$\lambda_1 = \lambda, \quad \lambda_2 = p_{12}\lambda, \quad \lambda_3 = p_{13}\lambda, \quad \lambda_4 = (1-p_{12}p_2)\lambda.$$

In the following it will be assumed that $\lambda_i/\mu_i < s_i$ for all i.

The arrival process at Q_1 is Poisson. By Burke's theorem, then, the equilibrium output from Q_1 is Poisson. The assignment of this output by lottery leads to a decomposition into independent Poisson streams with rates $\lambda_2 = p_{12}\lambda$ and $\lambda_3 = p_{13}\lambda$, respectively. We also note that the sum of two independent Poisson streams is Poisson. It can be concluded that the input to every queue is Poisson, so that each queue functions as an Erlang delay system with equilibrium state probabilities given by (4.3) and (4.4) of Chapter 3. Thus, for $i = 1,2,3,4$,

$$P_i(j_i) = \begin{cases} c_i \dfrac{(\lambda_i/\mu_i)^{j_i}}{j_i!} & (j_i = 0,\ldots,s_i-1), \\[2ex] c_i \dfrac{(\lambda_i/\mu_i)}{s_i!\, s_i^{\,j_i-s_i}} & (j_i = s_i, s_i+1, \ldots). \end{cases}$$

Furthermore, as a consequence of Burke's theorem, the states are independent, that is

$$P(j_1, j_2, j_3, j_4) = P_1(j_1)P_2(j_2)P_3(j_3)P_4(j_4).$$

$\boxed{b}$ With feedback from Q_2 to Q_1, the mean arrival rate at Q_1 is

$$\lambda_1^* = \lambda + (p_{12}p_2)\lambda + (p_{12}p_2)^2\lambda + \cdots$$

In this particular case, therefore, the mean arrival rates are

$$\lambda_1^* = \frac{\lambda}{1-p_{12}p_2}, \quad \lambda_2^* = \frac{p_{12}\lambda}{1-p_{12}p_2}, \quad \lambda_3^* = \frac{p_{13}\lambda}{1-p_{12}p_2}, \quad \lambda_4^* = \lambda.$$

Let $\mu_i(j)$ be defined as in (2.12). Then the equilibrium state equations are:

$$\begin{aligned} &(\lambda + \mu_1(j_1) + \mu_2(j_2) + \mu_3(j_3) + \mu_4(j_4))\, P(j_1, j_2, j_3, j_4) = \\ &\quad \lambda P(j_1-1, j_2, j_3, j_4) + \mu_1(j_1+1)\, p_{12} P(j_1+1, j_2-1, j_3, j_4) + \mu_1(j_1+1)\, p_{13} P(j_1+1, j_2, j_3-1, j_4) \\ &\quad + \mu_2(j_2+1)\, p_2 P(j_1-1, j_2+1, j_3, j_4) + \mu_2(j_2+1)\, p_{24} P(j_1, j_2+1, j_3, j_4-1) \\ &\quad + \mu_3(j_3+1) P(j_1, j_2, j_3+1, j_4-1) + \mu_4(j_4+1) P(j_1, j_2, j_3, j_4+1) \qquad (j_1, j_2, j_3, j_4 \geq 0) \end{aligned}$$

(Chap. 4, Ex. 8 b)

Consider the following five equations obtained by pairing terms on LHS and RHS of the equilibrium state equations,

$$\lambda P(j_1, j_2, j_3, j_4) = \mu_4(j_4+1) P(j_1, j_2, j_3, j_4+1), \tag{1}$$

$$\mu_1(j_1) P(j_1, j_2, j_3, j_4) = \lambda P(j_1-1, j_2, j_3, j_4) + \mu_2(j_2+1) p_2 P(j_1-1, j_2+1, j_3, j_4), \tag{2}$$

$$\mu_2(j_2) P(j_1, j_2, j_3, j_4) = \mu_1(j_1+1) p_{12} P(j_1+1, j_2-1, j_3, j_4), \tag{3}$$

$$\mu_3(j_3) P(j_1, j_2, j_3, j_4) = \mu_1(j_1+1) p_{13} P(j_1+1, j_2, j_3-1, j_4), \tag{4}$$

$$\mu_4(j_4) P(j_1, j_2, j_3, j_4) = \mu_2(j_2+1) p_{24} P(j_1, j_2+1, j_3, j_4-1) + \mu_3(j_3+1) P(j_1, j_2, j_3+1, j_4-1). \tag{5}$$

It is easily seen that a solution to (1)-(5) will also be a solution to the equilibrium state equations.

By (3), $\mu_2(j_2+1) P(j_1-1, j_2+1, j_3, j_4) = \mu_1(j_1) p_{12} P(j_1, j_2, j_3, j_4)$. Substitution into (2) and rewriting lead to

$$\frac{\lambda}{1-p_{12}p_2} P(j_1, j_2, j_3, j_4) = \mu_1(j_1+1) P(j_1+1, j_2, j_3, j_4). \tag{*}$$

A rewriting of (3) gives

$$\mu_2(j_2) = \mu_1(j_1+1) p_{12} \frac{P(j_1+1, j_2, j_3, j_4)}{P(j_1, j_2, j_3, j_4)} \frac{P(j_1+1, j_2-1, j_3, j_4)}{P(j_1+1, j_2, j_3, j_4)},$$

which, by (*), simplifies to

$$\mu_2(j_2) = \frac{p_{12}\lambda}{1-p_{12}p_2} \frac{P(j_1+1, j_2-1, j_3, j_4)}{P(j_1+1, j_2, j_3, j_4)}.$$

Hence,

$$\frac{p_{12}\lambda}{1-p_{12}p_2} P(j_1, j_2, j_3, j_4) = \mu_2(j_2+1) P(j_1, j_2+1, j_3, j_4). \tag{**}$$

Similarly, by (4) and (*),

$$\mu_3(j_3) = \frac{p_{13}\lambda}{1-p_{12}p_2} \frac{P(j_1+1, j_2, j_3-1, j_4)}{P(j_1+1, j_2, j_3, j_4)},$$

whereby

$$\frac{p_{13}\lambda}{1-p_{12}p_2} P(j_1, j_2, j_3, j_4) = \mu_3(j_3+1) P(j_1, j_2, j_3+1, j_4). \tag{***}$$

(Chap. 4, Ex. 8 b (cont'd))

Equations (*), (**), (***), have been derived from and are equivalent to Equations (2), (3), (4). Eq. (1) has the desired form, and we shall keep it the way it is. The final equation, (5), is redundant. To see this, combine (1) and (5) into

$$\lambda P(j_1, j_2, j_3, j_4) = \mu_2(j_2+1)\, p_{24} P(j_1, j_2+1, j_3, j_4) + \mu_3(j_3+1) P(j_1, j_2, j_3+1, j_4).$$

By (**) and (***), the right-hand side equals

$$\left[\frac{p_{12}\, p_{24}}{1-p_{12}p_2} + \frac{p_{13}}{1-p_{12}p_2}\right] \lambda P(j_1, j_2, j_3, j_4) = \lambda P(j_1, j_2, j_3, j_4)$$

since $p_{12}p_{24} + p_{13} = 1 - p_{12}p_2$. This proves redundancy.

We conclude that (1)-(5) are equivalent to the following system of equations,

$$\lambda_1^* P(j_1, j_2, j_3, j_4) = \mu_1(j_1+1)\, P(j_1+1, j_2, j_3, j_4), \qquad (6)$$

$$\lambda_2^* P(j_1, j_2, j_3, j_4) = \mu_2(j_2+1)\, P(j_1, j_2+1, j_3, j_4), \qquad (7)$$

$$\lambda_3^* P(j_1, j_2, j_3, j_4) = \mu_3(j_3+1) P(j_1, j_2, j_3+1, j_4), \qquad (8)$$

$$\lambda_4^* P(j_1, j_2, j_3, j_4) = \mu_4(j_4+1) P(j_1, j_2, j_3, j_4+1). \qquad (9)$$

where λ_i^* $(i=1,2,3,4)$ is the mean arrival rate at Q_i.

Recursive solution of Eq. (6), for example, gives, for fixed j_2, j_3, j_4,

$$P(j_1, j_2, j_3, j_4) = \begin{cases} P(0, j_2, j_3, j_4)\, \dfrac{(\lambda_1^*/\mu_1)^{j_1}}{j_1!} & (j_1 = 0, \dots, s_1-1), \\[2ex] P(0, j_2, j_3, j_4)\, \dfrac{(\lambda_1^*/\mu_1)^{j_1}}{s_1!\, s_1^{j_1 - s_1}} & (j_1 = s_1, s_1+1, \dots). \end{cases}$$

The marginal probability of j_1, $P_1(j_1)$, is found by summation over all possible j_2, j_3, j_4. In general we find

$$P_i(j_i) = \begin{cases} P_i(0)\, \dfrac{(\lambda_i^*/\mu_i)^{j_i}}{j_i!} & (j_i = 0, \dots, s_i - 1), \\[2ex] P_i(0)\, \dfrac{(\lambda_i^*/\mu_i)^{j_i}}{s_i!\, s_i^{j_i - s_i}} & (j_i = s_i, s_i+1, \dots). \end{cases}$$

In addition it may be shown that the condition for independence holds:

$$P(j_1, j_2, j_3, j_4) = P_1(j_1)\, P_2(j_2)\, P_3(j_3)\, P_4(j_4) \quad (j_i \geq 0,\ i = 1,2,3,4) \qquad \square$$

Chapter 4, Exercise 9

'Closed networks of queues.'

$\boxed{a}$ As before, let $\mu_i(j_i) = j_i\mu_i$ if $j_i \leq s_i$, $\mu_i(j_i) = s_i\mu_i$ if $j_i > s_i$. The equilibrium state equations are as follows,

$$\begin{aligned}
&(\mu_1(j_1) + \mu_2(j_2) + \cdots + \mu_m(j_m))\,P(j_1, j_2, \ldots, j_m) = \\
&\mu_1(j_1+1)\,P(j_1+1, j_2-1, j_3, \ldots, j_m) + \mu_2(j_2+1)\,P(j_1, j_2+1, j_3-1, \ldots, j_m) \\
&+ \cdots + \mu_m(j_m+1)\,P(j_1-1, j_2, \ldots, j_m+1) \qquad (j_i \geq 0 \text{ for } i = 1, \ldots, m;\ \Sigma j_i = n).
\end{aligned}$$

Consider the following m equations extracted from the equilibrium state equations,

$$\begin{aligned}
\mu_1(j_1)P(j_1, j_2, \ldots, j_m) &= \mu_m(j_m+1)\,P(j_1-1, j_2, \ldots, j_m+1), && (1') \\
\mu_2(j_2)P(j_1, j_2, \ldots, j_m) &= \mu_1(j_1+1)\,P(j_1+1, j_2-1, \ldots, j_m), && (2') \\
\mu_3(j_3)P(j_1, j_2, \ldots, j_m) &= \mu_2(j_2+1)\,P(j_1, j_2+1, j_3-1, \ldots, j_m), && (3') \\
&\ \vdots \\
\mu_{m-1}(j_{m-1})P(j_1, j_2, \ldots, j_m) &= \mu_{m-2}(j_{m-2}+1)\,P(\ldots, j_{m-2}+1, j_{m-1}-1, j_m), && ((m-1)') \\
\mu_m(j_m)P(j_1, j_2, \ldots, j_m) &= \mu_{m-1}(j_{m-1}+1)\,P(j_1, \ldots, j_{m-1}+1, j_m-1). && (m')
\end{aligned}$$

It is clear that a solution to these equations will also be a solution to the equilibrium state equations. Anyone of the equations can be omitted. We choose to discard (1).

Now write

$$\begin{aligned}
P(j_1, j_2, \ldots, j_m) = P(n, 0, \ldots, 0) &\times \frac{P(j_1, n-j_1, 0, \ldots, 0)}{P(n, 0, \ldots, 0, 0, 0)} \times \frac{P(j_1, j_2, n-j_1-j_2, 0, \ldots, 0)}{P(j_1, n-j_1, 0, \ldots, 0, 0, 0)} \\
&\times \cdots \times \frac{P(j_1, j_2, \ldots, j_m)}{P(j_1, j_2, \ldots, n-j_1-j_2-\cdots-j_{m-2}, 0)}.
\end{aligned}$$

By a similar rewriting and the application of Eq. (2′), factor no. 2 on the right can be expressed

$$\begin{aligned}
\frac{P(j_1, n-j_1, 0, \ldots, 0)}{P(n, 0, \ldots, 0, 0, 0)} &= \frac{P(n-1, 1, 0, \ldots, 0)}{P(n, 0, \ldots, 0, 0)} \times \frac{P(n-2, 2, 0, \ldots, 0)}{P(n-1, 1, 0, \ldots, 0)} \times \cdots \times \frac{P(j_1, n-j_1, 0, \ldots, 0)}{P(j_1+1, n-j_1-1, 0, \ldots, 0)} \\
&= \frac{\mu_1(n)}{\mu_2(1)} \times \frac{\mu_1(n-1)}{\mu_2(2)} \times \cdots \times \frac{\mu_1(j_1+1)}{\mu_2(n-j_1)}. \qquad (j_1 < n)
\end{aligned}$$

(Chap. 4, Ex. 9a)

Proceeding in this manner, using Eq. (k′) for factoring factor no. k, we deduce

$$P(j_1, j_2, \ldots, j_m) = P(n, 0, \ldots, 0)\, A_2 A_3 \cdots A_m, \qquad (*)$$

where, for $k = 2, 3, \ldots, m$,

$$A_k = \begin{cases} 1 & \left(\sum_{i=k}^{m} j_i = 0\right), \\ \dfrac{\mu_{k-1}\left(\sum_{i=k-1}^{m} j_i\right)}{\mu_k(1)} \dfrac{\mu_{k-1}\left(\sum_{i=k-1}^{m} j_i - 1\right)}{\mu_k(2)} \cdots \dfrac{\mu_{k-1}(j_{k-1}+1)}{\mu_k\left(\sum_{i=k}^{m} j_i\right)} & \left(\sum_{i=k}^{m} j_i \geq 1\right). \end{cases}$$

Cancellation of factors in (*) results in

$$P(j_1, j_2, \ldots, j_m) = \left[P(n, 0, \ldots, 0) \prod_{r=1}^{n} \mu_1(r)\right] \prod_{i=1}^{m} Q_i(j_i) \qquad \begin{pmatrix} 0 \leq j_i \leq n, \\ \Sigma j_i = n \end{pmatrix},$$

where $Q_i(j_i) = 1$ if $j_i = 0$, $Q_i(j_i) = [\mu_i(1)\mu_i(2) \cdots \mu_i(j_i)]^{-1}$ if $j_i \geq 1$. Finally, using $\Sigma P(j_1, j_2, \ldots, j_m) = 1$ and substituting the expression for $\mu_i(j)$, we obtain

$$P(j_1, j_2, \ldots, j_m) = \frac{\prod_{i=1}^{m} Q_i(j_i)}{\sum_S \prod_{i=1}^{m} Q_i(r_i)} \qquad \begin{pmatrix} 0 \leq j_i \leq n, \\ \Sigma j_i = n \end{pmatrix}, \qquad (1)$$

where $S = \{(r_1, r_2, \ldots, r_m) : 0 \leq r_i \leq n, \Sigma r_i = n\}$ and

$$Q_i(j_i) = \begin{cases} \dfrac{(1/\mu_i)^{j_i}}{j_i!} & (j_i < s_i), \\ \dfrac{(1/\mu_i)^{j_i}}{s_i!\, s_i^{\,j_i - s_i}} & (j_i \geq s_i). \end{cases}$$

The probability $P(j_1, j_2, \ldots, j_m)$ has been written as a product of factors dependent on $j_1, j_2, \ldots, j_m$, respectively, but the random variables $N_1, N_2, \ldots, N_m$ are not independent. The reason is that the set S for which Eq. (1) applies is not a product space $J_1 \times J_2 \times \cdots \times J_m$. If, for instance, $N_1 = n$, then $N_2 = 0$, so that, obviously, N_1 and N_2 cannot be independent variables.

(Chap. 4, Ex. 9 b)

$\boxed{b}$ In the general model, where a departure from Q_i with probability p_{ij} goes to Q_j $(i,j=1,2,\dots,m)$, the equilibrium state equations are

$$\sum_{i=1}^{m} \mu_i(j_i) P(j_1,\dots,j_m) = \sum_{i=1}^{m} \sum_{\substack{k=1 \\ k\neq i}}^{m} \mu_k(j_k+1) p_{ki} P(j_1,\dots,j_i-1,\dots,j_k+1,\dots,j_m) + \sum_{i=1}^{m} \mu_i(j_i) p_{ii} P(j_1,\dots,j_m). \qquad (2)$$

We shall verify that the solution is of the form (1), that is $P(j_1,\dots,j_m) = \prod_{i=1}^{m} Q_i(j_i) / \Sigma_{\mathcal{S}} \prod_{i=1}^{m} Q_i(r_i)$, but where, for some $\{x_i\}$,

$$Q_i(j_i) = \begin{cases} \dfrac{(1/x_i)^{j_i}}{j_i!} & (j_i < s_i), \\ \dfrac{(1/x_i)^{j_i}}{s_i!\, s_i^{\,j_i - s_i}} & (j_i \geq s_i). \end{cases} \qquad (3)$$

Substitution of (1) into (2) gives

$$\Big(\prod_{i=1}^{m} Q_i(j_i)\Big) \sum_{i=1}^{m} \mu_i(j_i) = \Big(\prod_{i=1}^{m} Q_i(j_i)\Big) \sum_{i=1}^{m} \sum_{\substack{k=1 \\ k\neq i}}^{m} \mu_k(j_k+1) p_{ki} \frac{Q_i(j_i-1)}{Q_i(j_i)} \frac{Q_k(j_k+1)}{Q_k(j_k)} + \Big(\prod_{i=1}^{m} Q_i(j_i)\Big) \sum_{i=1}^{m} \mu_i(j_i) p_{ii}.$$

Cancellation of the common factor $\prod_{i=1}^{m} Q_i(j_i)$ and the use of (3) results in

$$\sum_{i=1}^{m} \mu_i(j_i) = \sum_{i=1}^{m} \sum_{\substack{k=1 \\ k\neq i}}^{m} \mu_k(j_k+1) p_{ki} \frac{\mu_i(j_i) x_i}{\mu_i} \frac{\mu_k}{\mu_k(j_k+1) x_k} + \sum_{i=1}^{m} \mu_i(j_i) p_{ii}.$$

Hence,

$$\sum_{i=1}^{m} \mu_i(j_i) \Big[1 - \mu_i^{-1} x_i \sum_{\substack{k=1 \\ k\neq i}}^{m} \mu_k x_k^{-1} p_{ki} - p_{ii} \Big] = 0,$$

whereby

$$\sum_{i=1}^{m} \mu_i(j_i) \Big[1 - \mu_i^{-1} x_i \sum_{k=1}^{m} \mu_k x_k^{-1} p_{ki} \Big] = 0. \qquad (4)$$

This leads to the requirement that

$$\sum_{k=1}^{m} p_{ki}(\mu_k x_k^{-1}) = \mu_i x_i^{-1} \qquad (i = 1,2,\dots,m). \qquad (5)$$

A solution $\{\mu_i x_i^{-1}\}$ $(\{x_i\})$ exists provided all queues "communicate." $\square$

Chapter 4, Exercise 10

'The following model can be used ...'

OBS! The server groups have been renamed : $G_1 \to H_2$, $G_2 \to H_1$.

For group H_1 (digit trunks for dialing) let s_1 be the number of (exponential) servers with service rate μ_1, and for group H_2 (time slots for talking) let s_2 be the number of (exponential) servers with service rate μ_2. The possibility that all servers in H_1 are busy, while a server in H_2 is idle, implies $s_1 < s_2$, since a call holding a server in H_1 will at the same time hold a server in H_2. Let j_1 = calls in dialing phase or in waiting position, and let j_2 = calls in talking phase. Let $\mu_1(j_1) = j_1\mu_1$ if $j_1 < s_1$, $\mu_1(j_1) = s_1\mu_1$ if $j_1 \geq s_1$.

The equilibrium state equations are, for $j_1, j_2 \geq 0$,

$$(\lambda + \mu_1(j_1) + j_2\mu_2)\,P(j_1,j_2) = \qquad (0 \leq j_1 + j_2 < s_2)$$
$$\lambda P(j_1-1,j_2) + \mu_1(j_1+1)P(j_1+1,j_2-1) + (j_2+1)\mu_2 P(j_1,j_2+1),$$

$$(\mu_1(j_1) + j_2\mu_2)\,P(j_1,j_2) = \qquad (j_1 + j_2 = s_2)$$
$$\lambda P(j_1-1,j_2) + \mu_1(j_1+1)P(j_1+1,j_2-1).$$

Hence the equations

$$\lambda P(j_1,j_2) = (j_2+1)\mu_2 P(j_1,j_2+1) \qquad (0 \leq j_1+j_2 < s_2),$$
$$\mu_1(j_1)P(j_1,j_2) = \lambda P(j_1-1,j_2) \qquad (0 \leq j_1+j_2 \leq s_2),$$
$$j_2\mu_2 P(j_1,j_2) = \mu_1(j_1+1)P(j_1+1,j_2-1) \qquad (0 \leq j_1+j_2 \leq s_2),$$

a solution to which will also solve the equilibrium state equations. Disregarding the last, redundant equation, solving the other two recursively starting with $P(0,0)$, we find

$$P(j_1,j_2) = c\,P_1(j_1)\,P_2(j_2) \qquad (0 \leq j_1+j_2 \leq s_2),$$

where

$$P_1(j_1) = \begin{cases} \dfrac{(\lambda/\mu_1)^{j_1}}{j_1!} & (j_1 < s_1), \\[2ex] \dfrac{(\lambda/\mu_1)^{j_1}}{s_1!\,s_1^{\,j_1-s_1}} & (s_1 \leq j_1 \leq s_2); \end{cases} \qquad P_2(j_2) = \frac{(\lambda/\mu_2)^{j_2}}{j_2!} \quad (0 \leq j_2 \leq s_2).$$

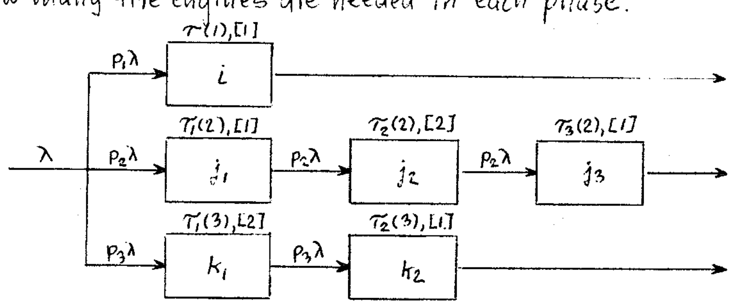
The constant c is determined by $\sum_{0 \leq j_1+j_2 \leq s_2} P(j_1,j_2) = 1$. □

Chapter 4, Exercise 11

'The following is a simplified version of a model...'

In the present model, the customers are the fires of any of three types, going through phases whose duration has an exponential distribution. The overall arrival rate is λ, and with probability p_r ($r = 1,2,3$) a fire is of type r. A fire of type 1 is characterized by a single phase with mean $\tau(1)$, requiring the use of 1 fire engine. A fire of type 2 goes through three phases with means $\tau_1(2), \tau_2(2), \tau_3(2)$, requiring 1, 2 and 1 fire engine(s), respectively. A fire of type 3 has two phases with means $\tau_1(3)$ and $\tau_2(3)$, requiring 2 and 1 fire engines respectively.

The fires pass through a network of infinite-server queues. In the figure, the variables $i, j_1, j_2, j_3, k_1, k_2$ denote number of fires in progress, and the numbers in square brackets indicate how many fire engines are needed in each phase.



In reality, the network is composed of three independent queueing systems with arrival rates $p_1\lambda, p_2\lambda, p_3\lambda$, respectively. The first queueing system is an infinite-server queue, whose equilibrium distribution, by Eq. (4.27) of Chapter 2, is $P_i = ([p_1\lambda\tau(1)]^i/i!)\,e^{-p_1\lambda\tau(1)}$. The other two queueing systems are tandem queues. By Burke's theorem, the input to each constituent queue is Poisson, so that also here the state variables follow a Poisson distribution, and, furthermore, the equilibrium states of the queues are independent. Hence,

$$P(i; j_1, j_2, j_3; k_1, k_2) = \frac{[p_1\lambda\tau(1)]^i}{i!}\,\frac{[p_2\lambda\tau_1(2)]^{j_1}}{j_1!}\,\frac{[p_2\lambda\tau_2(2)]^{j_2}}{j_2!}\,\frac{[p_2\lambda\tau_3(2)]^{j_3}}{j_3!}\,\frac{[p_3\lambda\tau_1(3)]^{k_1}}{k_1!}\,\frac{[p_3\lambda\tau_2(3)]^{k_2}}{k_2!}\cdot c\,,$$

where,
$$c = \exp\{-[p_1\lambda\tau(1) + p_2\lambda\tau_1(2) + p_2\lambda\tau_2(2) + p_2\lambda\tau_3(2) + p_3\lambda\tau_1(3) + p_3\lambda\tau_2(3)]\}.$$

The distribution of $m = i + j_1 + 2j_2 + j_3 + 2k_1 + k_2$ is found by convoluting $i, j_1, 2j_2$ etc. $\square$

Chapter 4, Exercise 12

'Apportioning the moments of the overflow distribution.'

a The equilibrium state equations for the distribution $\{h(j,k_1,k_2)\}$ are

$$(a+j+k_1+k_2)\,h(j,k_1,k_2) = a\,h(j-1,k_1,k_2) + (j+1)h(j+1,k_1,k_2) + (k_1+1)h(j,k_1+1,k_2) + (k_2+1)h(j,k_1,k_2+1) \qquad \begin{pmatrix} j=0,1,\ldots,s-1; \\ k_1,k_2=0,1,\ldots \end{pmatrix}$$

$$(a+s+k_1+k_2)\,h(s,k_1,k_2) = a\,h(s-1,k_1,k_2) + a_1 h(s,k_1-1,k_2) + a_2 h(s,k_1,k_2-1) + (k_1+1)h(s,k_1+1,k_2) + (k_2+1)h(s,k_1,k_2+1) \qquad \begin{pmatrix} j=s; \\ k_1,k_2=0,1,\ldots \end{pmatrix}$$

We shall verify that

$$h(j,k_1,k_2) = P(j,k)\binom{k}{k_1}\left(\frac{a_1}{a}\right)^{k_1}\left(\frac{a_2}{a}\right)^{k_2}, \qquad (*)$$

where $k = k_1+k_2$. We begin by showing that the suggested solution satisfies the equilibrium state equations. Insertion of the above expression for $h(j,k_1,k_2)$ into the two sets of equilibrium state equations, and a straightforward reduction, result in Eqs. (3.3) and (3.4), respectively, which are always satisfied. Hence, the suggested solution is indeed the solution, at least up to a factor. Now, it is easily shown that $\Sigma_{k_1+k_2=k}\, h(j,k_1,k_2) = P(j,k)$ by the given expression, just as it should, so the expression does give the correct value.

b By definition of a conditional probability, $h(j,k_1,k_2)$ may be expressed as $h(j,k_1,k_2) = P(j,k)\,P\{N_1=k_1, N_2=k_2 \mid N=k\}$. By comparison with (*), therefore

$$P\{N_1=k_1, N_2=k_2 \mid N=k\} = \binom{k}{k_1}\left(\frac{a_1}{a}\right)^{k_1}\left(\frac{a_2}{a}\right)^{k_2}. \qquad (4)$$

c A binomial variable X with parameters (n,p) has mean $E(X) = np$ and variance $V(X) = np(1-p)$. Hence, $E(X^2) = V(X) + E^2(X) = n(n-1)p^2 + np$. By (4), for $N=k$, N_i is a binomial variable with $(n,p) = (k, a_i/a)$. It follows that

$$E(N_i \mid N=k) = k\,\frac{a_i}{a} \qquad (5)$$

$$E(N_i^2 \mid N=k) = k(k-1)\left(\frac{a_i}{a}\right)^2 + k\,\frac{a_i}{a} \qquad (6)$$

(Chap. 4, Ex. 12 d)

$\boxed{d}$ By (5), $E(N_i) = E_N(E(N_i|N=k)) = E_N(N\frac{a_i}{a})$. Hence,

$$E(N_i) = \frac{a_i}{a}E(N). \qquad (1)$$

$\boxed{e}$ By (6), $E(N_i^2) = E_N(E(N_i^2|N=k)) = E_N(N(N-1)(\frac{a_i}{a})^2 + N\frac{a_i}{a})$. Hence

$$E(N_i^2) = (\frac{a_i}{a})^2E(N^2) + \frac{a_i}{a}(1-\frac{a_i}{a})E(N). \qquad (7)$$

$\boxed{f}$ By (1) and (7), $V(N_i) = E(N_i^2) - E^2(N_i) = (\frac{a_i}{a})^2[E(N^2) - E^2(N)] + \frac{a_i}{a}(1-\frac{a_i}{a})E(N)$. Hence,

$$V(N_i) = (\frac{a_i}{a})^2V(N) + \frac{a_i}{a}(1-\frac{a_i}{a})E(N). \qquad (2)$$

Dividing Eq. (1) into Eq. (2), we get

$$\frac{V(N_i)}{E(N_i)} = \frac{a_i}{a}\frac{V(N)}{E(N)} + (1-\frac{a_i}{a}).$$

Letting $z = V(N)/E(N)$ and $z_i = V(N_i)/E(N_i)$, we find, for $p_i = a_i/a$,

$$z_i - 1 = p_i(z-1). \qquad (8)$$

$\boxed{g}$ By (5) and (6), when $n=2$,

$$\begin{aligned} E(N_1N_2|N=k) &= E(N_1(N-N_1)|N=k) \\ &= k\,E(N_1|N=k) - E(N_1^2|N=k) \\ &= k^2\frac{a_1}{a} - k(k-1)(\frac{a_1}{a})^2 - k\frac{a_1}{a} \\ &= \frac{a_1}{a}(1-\frac{a_1}{a})k(k-1). \end{aligned}$$

Hence, $E(N_1N_2) = E_N(E(N_1N_2|N=k)) = \frac{a_1}{a}(1-\frac{a_1}{a})E(N(N-1))$, or,

$$E(N_1N_2) = \frac{a_1}{a}\frac{a_2}{a}[E(N^2) - E(N)].$$

By (1),

$$E(N_1)E(N_2) = \frac{a_1}{a}\frac{a_2}{a}E^2(N).$$

Thus,

$$Cov(N_1,N_2) = E(N_1N_2) - E(N_1)E(N_2) = \frac{a_1}{a}\frac{a_2}{a}[V(N) - E(N)]. \qquad (3) \quad \square$$

Chapter 4, Exercise 13

'Show that when $a_1 = a_2 = \cdots = a_{s-1} = 0$ and $a_s = a$, then ...'

When $\lambda_1 = \lambda_2 = \cdots = \lambda_{s-1} = 0$ and $\lambda_s = \lambda$, then Eq. (4.6) becomes

$$\lambda_s P(j-1) = j\mu P(j) \qquad (j = 1, 2, \ldots, s-1),$$
$$\lambda_s P(s-1) + \lambda_s P(s) = s\mu P(s) \qquad (j = s),$$

or,

$$a P(j-1) = j P(j) \qquad (j = 1, 2, \ldots, s-1),$$
$$a P(s-1) = (s-a) P(s) \qquad (j = s).$$

The solution, in terms of $P(0)$, is

$$P(j) = \frac{a^j}{j!} P(0) \qquad (j = 0, 1, \ldots, s-1),$$
$$P(s) = \frac{a^s}{s!(1-a/s)} P(0).$$

Hence, and by Eq. (4.8) of Chapter 3, if $a < s$,

$$P(s) = \frac{\dfrac{a^s}{s!(1-a/s)}}{\sum_{k=0}^{s-1} \dfrac{a^k}{k!} + \dfrac{a^s}{s!(1-a/s)}} = C(s, a).$$

Chapter 4, Exercise 14

'Consider again the premise of Exercise 7...'

For a BCD queue with heterogeneous, exponential servers and random selection of server, let $P(x_1, \ldots, x_s; k) = P\{X_1 = x_1, \ldots, X_s = x_s; Q = k\}$.

When $\sum_{i=1}^{s} x_i < s$, the equilibrium state equations are precisely as in the BCC queue of Exercise 7, with $P(x_1, \ldots, x_s; 0)$ replacing $\tilde{P}(x_1, \ldots, x_s)$.

When $\sum_{i=1}^{s} x_i = s$ ($x_i = 1$ for all i) and $k = 0$, we now have

$$\left(\lambda + \sum_{i=1}^{s} x_i \mu_i\right) P(x_1, \ldots, x_s; 0) = \qquad \left(\sum_{i=1}^{s} x_i = s\right)$$
$$\left(\sum_{i=1}^{s} \mu_i\right) P(x_1, \ldots, x_s; 1) \qquad (*)$$
$$+ \lambda\left(P(x_1 - 1, x_2, \ldots, x_s; 0) + P(x_1, x_2 - 1, x_3, \ldots, x_s; 0) + \cdots + P(x_1, \ldots, x_{s-1}, x_s - 1; 0)\right).$$

(Chap. 4, Ex. 14)

In addition we have the rate up = rate down equations

$$\lambda P(1,\dots,1;k) = \left(\sum_{i=1}^{s}\mu_i\right) P(1,\dots,1;k+1) \qquad (k=0,1,\dots). \qquad (**)$$

By subtraction of Eq. (**), for $k=0$, from Eq. (*), we derive

$$\left(\sum_{i=1}^{s} x_i\mu_i\right) P(x_1,\dots,x_s;0) = \qquad \left(\sum_{i=1}^{s} x_i = s\right)$$
$$\lambda\left(P(x_1-1,x_2,\dots,x_s;0) + P(x_1,x_2-1,x_3,\dots,x_s;0) + P(x_1,\dots,x_{s-1};x_s-1;0)\right). \qquad (***)$$

Observe that (***) is the remaining equilibrium state equation in the BCC case of Exercise 7, with $P(x_1,\dots,x_s;0)$ instead of $\tilde{P}(x_1,\dots,x_s)$.

We conclude that for $k=0$ the solution is the same as in the BCC case except for a proportionality constant,

$$P(x_1,\dots,x_s;0) = c\,\tilde{P}(x_1,\dots,x_s) \qquad (x_i=0,1;\ i=1,\dots,s). \qquad (1)$$

By (**),

$$P(1,\dots,1;k) = \left(\frac{\lambda}{\sum_{i=1}^{s}\mu_i}\right)^k P(1,\dots,1;0) \qquad (k=1,2,\dots).$$

Utilization of (1) and the definition $\rho = \lambda/\sum_{i=1}^{s}\mu_i$ (utilization factor) give

$$P(1,\dots,1;k) = c\rho^k\,\tilde{P}(1,\dots,1) \qquad (k=1,2,\dots). \qquad (2)$$

Substitution into the normalization equation leads to

$$\begin{aligned} 1 &= \sum_{0\le\Sigma x_i\le s} P(x_1,\dots,x_s;0) + \sum_{k=1}^{\infty} P(1,\dots,1;k) \\ &= c\sum_{0\le\Sigma x_i\le s}\tilde{P}(x_1,\dots,x_s) + c\,\tilde{P}(1,\dots,1)\sum_{k=1}^{\infty}\rho^k \\ &= c + c\,\tilde{P}(1,\dots,1)\frac{\rho}{1-\rho} \end{aligned}$$

Hence,

$$c = \left(1 + \frac{\rho}{1-\rho}\tilde{P}(1,\dots,1)\right)^{-1} \qquad (3)$$

Thus, the equilibrium probabilities for the BCD queue are given by (1), (2), and (3), where $\{\tilde{P}(x_1,\dots,x_s)\}$ are the equilibrium probabilities of the corresponding BCC queue of Exercise 7. □

Chapter 4, Exercise 15

'Show that, if $a>0$, then $\vartheta=\alpha$ when $s=0$, $\vartheta>\alpha$ when $s\geq 1$...'

$$\alpha = aB(s,a), \qquad (3.1)$$

$$\vartheta = \alpha\left(1-\alpha+\frac{a}{s+1+\alpha-a}\right). \qquad (3.2)$$

$s=0$. Clearly, $B(0,a)=1$. By (3.1) and (3.2) then, $\vartheta=\alpha=a$. This should come as no surprise, as the equilibrium state of an infinite-server system with Poisson input has the Poisson distribution with parameter (mean and variance) equal to a.

$s\geq 1$. By Exercise 6 of Chapter 3,

$$B(s,a)=\frac{aB(s-1,a)}{s+aB(s-1,a)} \qquad (s\geq 1).$$

Hence follow the two equivalent equations

$$s+1+aB(s,a)=\frac{aB(s,a)}{B(s+1,a)} \qquad (s\geq 0), \qquad (*)$$

$$\frac{B(s+1,a)}{B(s,a)}=\frac{a(1-B(s+1,a))}{s+1} \qquad (s\geq 0). \qquad (**)$$

Now, for $s\geq 1$,

$$\begin{aligned}
\vartheta>\alpha \;&\Leftrightarrow\; \frac{a}{s+1+\alpha-a}>\alpha && [\text{by } (3.2)]\\
&\Leftrightarrow\; \frac{a}{s+1+aB(s,a)-a}>aB(s,a) && [\text{by } (3.1)]\\
&\Leftrightarrow\; \frac{B(s+1,a)}{B(s,a)}>a[B(s,a)-B(s+1,a)] && [\text{by } (*)]\\
&\Leftrightarrow\; \frac{a[1-B(s+1,a)]}{s+1}>a[B(s,a)-B(s+1,a)]. && [\text{by } (**)]
\end{aligned}$$

As usual, let $\tilde{\rho}_j$ denote the load on the j'th ordered server in an Erlang loss system. By Eq. (3.18) of Chapter 3, $\tilde{\rho}_{s+1}=a[B(s,a)-B(s+1,a)]$ and $\sum_{j=1}^{s+1}\tilde{\rho}_j=a[1-B(s+1,a)]$ (= carried load with $s+1$ servers). Thus, for $s\geq 1$,

$$\vartheta>\alpha \;\Leftrightarrow\; \sum_{j=1}^{s+1}\tilde{\rho}_j/(s+1) > \tilde{\rho}_{s+1}.$$

According to Messerli [1972], cf. Sec. 3 of Chap. 3, $\tilde{\rho}_1>\tilde{\rho}_2>\tilde{\rho}_3\cdots$. Hence, $s\geq 1 \Rightarrow \sum_{j=1}^{s+1}\tilde{\rho}_j/(s+1)>\tilde{\rho}_s$. The conclusion then is

$$s\geq 1 \Rightarrow \vartheta>\alpha. \qquad \square$$

Chapter 4, Exercise 16

'a_1 = 10 erl of Poisson traffic is offered to a group of 10 servers...'

The exercise requires the use of the Equivalent Random Method. The case is described by Figure 4-8 for $n = 2$. Offered loads and server group sizes are:

Parts (a) and (b): $(s_1, a_1) = (10, 10)$, $(s_2, a_2) = (5, 5)$, c to be calculated.
Part (c): $(s_1, a_1^*) = (10, 15)$, $(s_2, a_2^*) = (5, 7.5)$, c as in (a) & (b).

$\boxed{a}$ The values of $B(s_1, a_1)$ and $B(s_2, a_2)$ may be read off Figure A-1 in the Appendix. Exact values obtained from tables of the Poisson distribution or the Erlang B-formula are

$$B(s_1, a_1) = B(10, 10) = 0.215\_ ,$$
$$B(s_2, a_2) = B(5, 5) = 0.285\_ .$$

For the two primary groups, the mean and the variance of the equilibrium state of an infinite-server backup group are, by Eqs. (3.1) and (3.2),

$$\alpha_1 = 2.15, \qquad \vartheta_1 = 4.35,$$
$$\alpha_2 = 1.42, \qquad \vartheta_2 = 2.34.$$

Hence, the total overflow is characterized by the parameters

$$\alpha = \alpha_1 + \alpha_2 = 3.57,$$
$$\vartheta = \vartheta_1 + \vartheta_2 = 6.69,$$
$$z = \vartheta/\alpha = 1.87,$$

whereby

$$\tilde{a} = \vartheta + 3z(z-1) = 11.6 \qquad [\text{by } (7.15)],$$
$$\tilde{s} = \frac{\tilde{a}(\alpha+z)}{\alpha+z-1} - \alpha - 1 = 9.6 \qquad [\text{by } (7.16)].$$

Hence,

$$s = [\tilde{s}] = 9,$$
$$a = \frac{(s+\alpha+1)(\alpha+z-1)}{\alpha+z} = 11.1 \qquad [\text{by } (7.17)].$$

Thus, $(s, a) = (9, 11.1)$ define the equivalent random system having the approximate overflow characteristics α and ϑ of the total overflow from the two primary groups.

(Chap. 4, Ex. 16 a)

When the overflow group has size c, the loss among the overflow customers (from the equivalent random group) is, by (7.13),

$$\Pi_c = \frac{B(s+c,a)}{B(s,a)} = \frac{B(9+c,11.1)}{B(9,11.1)}.$$

By Figure A-1, $B(9,11.1) = 0.323$. Thus the requirement $\Pi_c < 0.10$ translates into the condition

$$B(9+c,11.1) < 0.0323 < B(9+c-1,11.1).$$

From Figure A-1 we obtain

$$B(17,11.1) = 0.0259, \qquad B(16,11.1) = 0.0408.$$

Hence, our estimate of the necessary capacity, i.e. size, of the overflow group becomes

$$c = 17 - s = \underline{8}, \tag{1}$$

corresponding to an estimated loss on the overflow customers equal to

$$\Pi_c = \frac{B(17,11.1)}{B(9,11.1)} = \frac{0.0259}{0.323} = \underline{0.080}. \tag{2}$$

$\boxed{b}$ The estimate of the loss for the system as a whole is, by (7.14),

$$\Pi = \frac{aB(s+c,a)}{a_1+a_2} = \frac{11.1\,B(17,11.1)}{10+5} = \frac{0.287}{15} = \underline{0.019}. \tag{3}$$

$\boxed{c}$ After increasing the loads on the primary groups by 50% the new offered loads will be $a_1^* = 15$ and $a_2^* = 7.5$. We shall estimate the losses on the overflow group as well as the system as a whole assuming the old server group sizes, $s_1 = 10$, $s_2 = 5$, and $c = 8$. To begin, the losses on the primary groups are, by Fig. A-1,

$$B(s_1,a_1^*) = B(10,15) = 0.410,$$
$$B(s_2,a_2^*) = B(5,7.5) = 0.453.$$

(Chap. 4, Ex. 16 c)

The calculation of an approximate equivalent random system proceeds along the same lines as in part (a). The results are as follows,

$$\alpha_1^* = 6.15, \qquad \vartheta_1^* = 11.23,$$
$$\alpha_2^* = 3.40, \qquad \vartheta_2^* = 5.26,$$

$$\alpha^* = \alpha_1^* + \alpha_2^* = 9.55,$$
$$\vartheta^* = \vartheta_1^* + \vartheta_2^* = 16.49,$$
$$z^* = \vartheta^*/\alpha^* = 1.73,$$

$$\tilde{a}^* = \vartheta^* + 3z^*(z^*-1) = 20.3,$$
$$\tilde{s}^* = \frac{\tilde{a}^*(\alpha^*+z^*)}{\alpha^*+z^*-1} - \alpha^* - 1 = 11.7,$$

$$s^* = [\tilde{s}^*] = 11,$$
$$a^* = \frac{(s^*+\alpha^*+1)(\alpha^*+z^*-1)}{\alpha^*+z^*} = 19.6.$$

Thus, $(s^*, a^*) = (11, 19.6)$ define the equivalent random system having the approximate overflow characteristics α^* and ϑ^* of the total overflow from the two primary groups, after the 50% increase in loads.

The estimated loss on the overflow group is now calculated to, by the use of Figure A-2 in the Appendix,

$$\Pi_c^* = \frac{B(s^*+c, a^*)}{B(s^*, a^*)} = \frac{B(19, 19.6)}{B(11, 19.6)}$$
$$= \frac{0.179}{0.485}$$
$$= \underline{0.369}, \qquad (4)$$

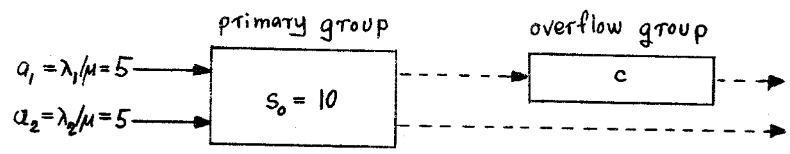
and the estimated loss for the system as a whole is

$$\Pi^* = \frac{a^* B(s^*+c, a^*)}{a_1^* + a_2^*} = \frac{19.6\, B(19, 19.6)}{15 + 7.5}$$
$$= \frac{3.51}{22.5}$$
$$= \underline{0.156}. \qquad (5)$$

□

Chapter 4, Exercise 17

'Poisson traffic totaling 10 erl is offered to a group of 10 servers.'



Total overflow

As in a similar case in Exercise 16, the total overflow from the primary group is characterized by the mean and the variance of the equilibrium state of a fictitious infinite-server back-up group, equal to

$$\alpha = (a_1+a_2)\,B(s_0, a_1+a_2) \qquad = 2.15,$$

$$\vartheta = \alpha\left(1-\alpha + \frac{a_1+a_2}{s_0+1+\alpha-(a_1+a_2)}\right) \qquad = 4.35.$$

High-priority overflow

By Exercise 12, mean and variance of the high-priority overflow stream are, respectively,

$$\alpha_1 = \frac{a_1}{a_1+a_2}\alpha = a_1 B(s_0, a_1+a_2) \qquad = 1.07,$$

$$\vartheta_1 = \left(\frac{a_1}{a_1+a_2}\right)^2\vartheta + \frac{a_1}{a_1+a_2}\left(1-\frac{a_1}{a_1+a_2}\right)\alpha \qquad = 1.62.$$

Equivalent random system

The decision on c will be based upon a calculation of an equivalent random system (s,a) whose overflow has approximately the mean and the variance of the overflow stream of high-priority customers from the primary group. First, we calculate

$$z_1 = \vartheta_1/\alpha_1 = 1.51.$$

(Chap. 4, Ex. 17)

As a first approximation we calculate

$$\tilde{a} = \vartheta_1 + 3z_1(z_1-1) = 3.93 \qquad [\text{by } (7.15)],$$

$$\tilde{s} = \frac{\tilde{a}(\alpha_1+z_1)}{\alpha_1+z_1-1} - \alpha_1 - 1 = 4.35 \qquad [\text{by } (7.16)].$$

Hence, by (7.17),

$$a = \frac{([\tilde{s}]+\alpha_1+1)(\alpha_1+z_1-1)}{\alpha_1+z_1} = 3.72,$$

so that the equivalent random system is described by

$$(s, a) = (4, 3.72).$$

Calculation of c

Our estimate of the loss of high-priority customers on the system as a whole is

$$\Pi = \frac{aB(s+c,a)}{\alpha_1} = \frac{3.72\,B(4+c,3.72)}{5}.$$

The smallest c meeting the requirement $\Pi < 0.01$ therefore must satisfy the inequalities

$$B(4+c, 3.72) < 0.0134 < B(4+c-1, 3.72)$$

By Figure A-1,

$$B(9, 3.72) = 0.0092, \qquad B(8, 3.72) = 0.0223.$$

It follows that the size of the overflow group should be

$$c = 9 - s = 5,$$

for which

$$\Pi = 0.0068.$$

$\square$

Chapter 4, Exercise 18

'Consider the Erlang loss system with hyperexponential service times.'

Let j_1 be the number of customers whose service time is exponential with mean μ_1^{-1} (with probability p_1), and let j_2 be the number of customers whose service time is exponential with mean μ_2^{-1} (with probability p_2). Let $P(j_1, j_2)$ denote the equilibrium probability that the state of the system is (j_1, j_2). The conservation-of-flow equations are

$$\begin{aligned}(\lambda p_1 + \lambda p_2 + j_1\mu_1 + j_2\mu_2) P(j_1, j_2) &= \qquad\qquad (0 \le j_1 + j_2 < s)\\ &\lambda p_1 P(j_1 - 1, j_2) + \lambda p_2 P(j_1, j_2 - 1)\\ &+ (j_1 + 1)\mu_1 P(j_1 + 1, j_2) + (j_2 + 1)\mu_2 P(j_1, j_2 + 1)\end{aligned}$$

$$(j_1\mu_1 + j_2\mu_2) P(j_1, j_2) = \lambda p_1 P(j_1 - 1, j_2) + \lambda p_2 P(j_1, j_2 - 1) \qquad (j_1 + j_2 = s).$$

From these equations we extract the following two sets of equations,

$$\lambda p_1 P(j_1, j_2) = (j_1 + 1)\mu_1 P(j_1 + 1, j_2) \qquad (0 \le j_1 + j_2 < s),$$
$$\lambda p_2 P(j_1, j_2) = (j_2 + 1)\mu_2 P(j_1, j_2 + 1) \qquad (0 \le j_1 + j_2 < s).$$

The solution of the above equations, which also is a solution of the equilibrium state equations, is

$$P(j_1, j_2) = \frac{(\lambda p_1/\mu_1)^{j_1}}{j_1!} \frac{(\lambda p_2/\mu_2)^{j_2}}{j_2!} c \qquad (0 \le j_1 + j_2 \le s).$$

It follows that the equilibrium probability that altogether j $(= j_1 + j_2)$ customers will be in service equals

$$P_j = \sum_{j_1 + j_2 = j} P(j_1, j_2) = \frac{[(\lambda p_1/\mu_1) + (\lambda p_2/\mu_2)]^j}{j!} c \qquad (0 \le j \le s).$$

Introducing the unconditional mean service time by

$$\frac{1}{\mu} = \frac{p_1}{\mu_1} + \frac{p_2}{\mu_2},$$

it is readily verified that

$$P_j = \frac{(\lambda/\mu)^j/j!}{\sum_{k=0}^{s} (\lambda/\mu)^k/k!} \qquad (0 \le j \le s).$$

□

Chapter 4, Exercise 19

'Show that if a random variable X ...'

We need the fact that, if X_i is an exponentially distributed variable with parameter μ_i, that is $P\{X_i \leq t\} = 1 - e^{-\mu_i t}$, then

$$E(X_i) = \mu_i^{-1},$$
$$E(X_i^2) = 2\mu_i^{-2},$$
$$V(X_i) = E(X_i^2) - E^2(X_i) = \mu_i^{-2}.$$

Case 1: X is a sum of independent, exponential variables

Clearly,

$$E(X) = \Sigma_i E(X_i) = \Sigma_i \mu_i^{-1},$$
$$V(X) = \Sigma_i V(X_i) = \Sigma_i \mu_i^{-2}.$$

Hence,

$$E^2(X) = (\Sigma_i \mu_i^{-1})^2 = \Sigma_i \mu_i^{-2} + \sum_i \sum_{j \neq i} \mu_i^{-1}\mu_j^{-1} > \Sigma_i \mu_i^{-2} = V(X)$$

Since $E(X) > 0$, we can conclude that

$$E(X) > \sqrt{V(X)}. \qquad (1)$$

Case 2: X is a mixture of independent, exponential variables

Clearly,

$$E(X) = \Sigma_i p_i E(X_i) = \Sigma_i p_i \mu_i^{-1},$$
$$E(X^2) = \Sigma_i p_i E(X_i^2) = 2\Sigma_i p_i \mu_i^{-2},$$
$$V(X) = E(X^2) - E^2(X) = 2\Sigma_i p_i \mu_i^{-2} - (\Sigma_i p_i \mu_i^{-1})^2.$$

By Schwarz' inequality, $(\Sigma_i a_i b_i)^2 < \Sigma_i a_i^2 \, \Sigma_i b_i^2$. Hence,

$$(\Sigma_i p_i \mu_i^{-1})^2 = (\Sigma_i \sqrt{p_i}\sqrt{p_i}\mu_i^{-1})^2 < \Sigma_i p_i \mu_i^{-2},$$

whereby the inequality

$$V(X) - E^2(X) = 2[\Sigma_i p_i \mu_i^{-2} - (\Sigma_i p_i \mu_i^{-1})^2] > 0.$$

Thus, in this case,

$$E(X) < \sqrt{V(X)}. \qquad (2) \qquad \square$$