Chapter 2, Exercise 1

'In the model considered above, suppose that it costs c dollars...'

$$E\left(\frac{c}{Y}\right) = \sum_{j=1}^{\infty} \frac{c}{j} P\{Y=j\}$$
$$= \sum_{j=1}^{\infty} \frac{c}{j} \frac{j\,P\{X=j\}}{E(X)} \qquad \text{[by (1.5)]}$$
$$= \frac{c}{E(X)}.$$

Chapter 2, Exercise 2

'Consider a population modeled as a pure birth process...', cf. ex. 6

If $N(0)=0$, then, trivially, $P_0(t)=1$ for all t. Assume therefore $N(0)>0$. For notational convenience, let $n=N(0)$. The differential-difference equations (2.3) specialize to

$$\frac{d}{dt}P_j(t) = (j-1)\lambda P_{j-1}(t) - j\lambda P_j(t) \qquad (j=n, n+1, \ldots;\ P_{n-1}(t)=0)$$

with initial conditions (2.4): $P_n(0)=1$ and $P_j(0)=0$ for $j\neq n$. Differential-difference equations are given only for $j \geq n$ as evidently $P_j(t)=0$ for $t\geq 0$ when $j<n$.

In the case $n=1$, we have

$$\frac{d}{dt}P_j(t) = (j-1)\lambda P_{j-1}(t) - j\lambda P_j(t) \qquad (j=1,2,\ldots;\ P_0(t)=0)$$

with $P_1(0)=1$, $P_j(0)=0$ for $j\neq 1$.

For $j=1$, $\frac{d}{dt}P_1(t) = -\lambda P_1(t)$, so that

$$P_1(t) = e^{-\lambda t}.$$

Hence, $\frac{d}{dt}P_2(t) = \lambda e^{-\lambda t} - 2\lambda P_2(t)$. Applying standard methods in the solution of this linear differential equation one derives

$$P_2(t) = e^{-\lambda t}(1-e^{-\lambda t}).$$

The general formula, obtained by induction, is

$$P_j(t) = e^{-\lambda t}(1-e^{-\lambda t})^{j-1} \qquad (j=1,2,\ldots). \qquad \square$$

Chapter 2, Exercise 3

'Consider a birth-and-death process with $\mu_k = 0$ and $\mu_j > 0$ when $j > k$...'

First assume an initial state E_j with $j \geq k$. Then states E_j for $j = 0, 1, \dots, k-1$ are impossible. A relabeling of the states ($j' = j-k$) and an application of the above theorem, followed by a reverse relabeling, results in the stated equilibrium distribution $\{P_j\}$.

If, on the other hand, the initial state is E_j with $j < k$, then E_k will be reached eventually (with probability 1) and an application of the theorem leads, again, to the indicated limiting distribution. Thus, unconditionally, the equilibrium distribution is as stated for $S < \infty$. It also follows that $P_j = 0$ for $S = \infty$.

Chapter 2, Exercise 4

'Compound distributions.' — cf. Chap. 5, Ex. 5

[a] By the theorem of total probability,

$$P\{S_N = k\} = \sum_{n=0}^{\infty} P\{N = n\} P\{\sum_{j=1}^{n} X_j = k\}.$$

The probability generating function of S_N is

$$\begin{aligned} h(z) &= \sum_{k=0}^{\infty} P\{S_N = k\} z^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} P\{N=n\} P\{\sum_{j=1}^{n} X_j = k\} z^k \\ &= \sum_{n=0}^{\infty} P\{N=n\} \sum_{k=0}^{\infty} P\{\sum_{j=1}^{n} X_j = k\} z^k \\ &= \sum_{n=0}^{\infty} P\{N=n\}[f(z)]^n = g(f(z)). \end{aligned}$$

[b] Differentiating $h(z)$ twice,

$$h'(z) = g'(f(z)) \cdot f'(z),$$

$$h''(z) = g'(f(z)) f''(z) + g''(f(z))[f'(z)]^2.$$

Hence,

$$h'(1) = g'(f(1)) f'(1) = g'(1) f'(1),$$

$$h''(1) = g'(f(1)) f''(1) + g''(f(1))[f'(1)]^2 = g'(1) f''(1) + g''(1)[f'(1)]^2.$$

(Chap. 2, Ex. 4 b)

By (4.5),

$$E(S_N) = h'(1) = g'(1)f'(1) = E(N)E(X).$$

By (4.5) and (4.8),

$$\begin{aligned} V(S_N) &= h''(1) + h'(1) - [h'(1)]^2 \\ &= g'(1)f''(1) + g''(1)[f'(1)]^2 + g'(1)f'(1) - [g'(1)]^2[f'(1)]^2 \\ &= g'(1)\left(f''(1) + f'(1) - [f'(1)]^2\right) + \left(g''(1) + g'(1) - [g'(1)]^2\right)[f'(1)]^2 \\ &= E(N)V(X) + V(N)E^2(X) \end{aligned}$$

Chapter 2, Exercise 5

'Let N_1 and N_2 be...' — cf. Ex. 23

Under procedure (a) each of the N_1 balls will be left unmarked with probability x, so by its definition $g(x)$ is the probability that none of the N_1 balls is marked. Similarly, $g(y)$ is the probability that none of the N_2 balls is marked. Hence, under procedure (a), $g(x)g(y)$ is the probability that none of the N_1+N_2 balls will be marked, provided that a ball placed in cell 1 (2) is left unmarked with probability x (y).

Under procedure (b) a ball from either batch will be left unmarked with probability $\frac{1}{2}x + \frac{1}{2}y$. It follows that the probability that none of the N_ν ($\nu = 1,2$) balls is marked is $g(\frac{x+y}{2})$. Hence, under procedure (b), $g(\frac{x+y}{2})g(\frac{x+y}{2})$ is the probability that none of the N_1+N_2 balls will be marked.

We take equivalence to mean that the probability distribution of balls in the two cells is the same for both procedures. If the procedures are equivalent in this sense, then, whatever x and y, the probability that no ball is marked must be the same under both procedures. Thus equivalence implies

$$g(x)\,g(y) = g^2\left(\tfrac{x+y}{2}\right).$$

□

$\boxed{\text{Chapter 2, Exercise 6}}$

'For the model of Exercise 2, Section 2.2, define...' — cf. Ex. 8

$\boxed{a}$ For $n = N(0) = 1$, we have found $P_0(t) = 0$ and

$$\frac{d}{dt} P_j(t) = (j-1)\lambda P_{j-1}(t) - j\lambda P_j(t) \quad (j = 1, 2, \ldots).$$

Hence,

$$\sum_{j=1}^{\infty} \frac{d}{dt} P_j(t) z^j = \sum_{j=1}^{\infty} (j-1)\lambda P_{j-1}(t) z^j - \sum_{j=1}^{\infty} j\lambda P_j(t) z^j,$$

$$\begin{aligned}\frac{\partial}{\partial t} P(z,t) &= \lambda z^2 \sum_{j=2}^{\infty} (j-1) P_{j-1}(t) z^{j-2} - \lambda z \sum_{j=1}^{\infty} j P_j(t) z^{j-1} \\ &= (\lambda z^2 - \lambda z) \sum_{j=1}^{\infty} j P_j(t) z^{j-1} \\ &= \lambda z(z-1) \frac{\partial}{\partial z} P(z,t).\end{aligned}$$

$\boxed{b}$ We shall verify that the above partial differential equation as well as the initial condition $n = N(0) =$ are satisfied by

$$P(z,t) = \frac{z e^{-\lambda t}}{1 - z + z e^{-\lambda t}}.$$

Differentiation results in

$$\frac{\partial}{\partial z} P(z,t) = \frac{e^{-\lambda t}}{(1 - z + z e^{-\lambda t})^2},$$

$$\frac{\partial}{\partial t} P(z,t) = \lambda z(z-1) \frac{e^{-\lambda t}}{(1 - z + z e^{-\lambda t})^2}.$$

It is seen that the expression for $P(z,t)$ satisfies the partial differential equation derived in part (a). The initial condition $P_1(0) = 1$ translates into the requirement $P(z,0) = \sum_{j=0}^{\infty} P_j(0) z^j = z$, which sure enough is met by the proposed expression for $P(z,t)$.

in Exercise 2 it was found that if $n = N(0) = 1$, then

$$P_j(t) = \begin{cases} 0 & (j = 0), \\ e^{-\lambda t}(1 - e^{-\lambda t})^{j-1} & (j = 1, 2, \ldots). \end{cases}$$

To this probability distribution corresponds the generating function

(Chap. 2, Ex. 6 b)

$$g(z,t) = \sum_{j=0}^{\infty} P_j(t) z^j = \sum_{j=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{j-1} z^j$$
$$= z e^{-\lambda t} \sum_{j=0}^{\infty} [z(1-e^{-\lambda t})]^j = \frac{z e^{-\lambda t}}{1 - z + z e^{-\lambda t}}.$$

Since $g(z,t)$ is identical to the given $P(z,t)$ and because of the one-to-one correspondence between distribution and generating function, it is true that $P(z,t)$ generates the distribution found in Exercise 2.

$\boxed{c}$ In the general case $N(0) = n \geq 1$, it was found in Exercise 2 that $P_0(t) = P_1(t) = \dots = P_{n-1}(t) = 0$ for all t, and

$$\frac{d}{dt} P_j(t) = (j-1)\lambda P_{j-1}(t) - j\lambda P_j(t) \qquad (j = n, n+1, \dots).$$

In the same way as in part (a) for $n=1$, we find

$$\frac{\partial}{\partial t} P_n(z,t) = \lambda z(z-1) \frac{\partial}{\partial z} P_n(z,t),$$

where $P_n(z,t)$ is the generating function for $P_j(t)$, given that $P_n(0) = 1$. We shall verify that

$$P_n(z,t) = P^n(z,t).$$

First, we will show that the proposed solution satisfies the above partial differential equation. This follows from

$$\frac{\partial}{\partial z} P_n(z,t) = n P^{n-1}(z,t) \frac{\partial}{\partial z} P(z,t),$$
$$\frac{\partial}{\partial t} P_n(z,t) = n P^{n-1}(z,t) \frac{\partial}{\partial t} P(z,t),$$

and the previously verified result $\frac{\partial}{\partial t} P(z,t) = \lambda z(z-1) \frac{\partial}{\partial z} P(z,t)$.

The proposed solution also meets the initial condition. For $t = 0$, $P_n(z,0) = P^n(z,0) = z^n$. By definition, $P_n(z,0) = P_0(0) + P_1(0) z^1 + \dots + P_n(0) z^n + \dots$. Equating the coefficients of the two polynomials we see that $P_n(0) = 1$ and $P_j(0) = 0$ for $j \neq n$ as required.

We conclude that $P_n(z,t) = P^n(z,t)$ is the unique solution. The result, $P_n(z,t) = P^n(z,t)$, should not come as a surprise.

(Chap. 2, Ex. 6c)

Clearly, the process with $N(0) = n$ may be interpreted as the sum of n independent processes, each with $N(0) = 1$. That is, the state $N(t)$, given $N(0) = n$, equals the sum of the states $N_1(t), N_2(t), \ldots, N_n(t)$ of independent processes with $N_1(0) = \cdots = N_n(0) = 1$. By a fundamental property of generating functions, then $P_n(z,t) = P^n(z,t)$.

Chapter 2, Exercise 7

'Suppose S_{n_1} has the binomial distribution ...'

S_{n_1} is the sum of n_1 independent Bernoulli variables, and S_{n_2} is the sum of n_2 independent Bernoulli variables, all of which are independent and have parameter p. Hence, the sum $S_{n_1} + S_{n_2}$ is the sum of $n_1 + n_2$ independent Bernoulli variables with parameter p. That is, $S_{n_1} + S_{n_2}$ has the binomial distribution (5.1) with $n = n_1 + n_2$.

Alternatively, the generating functions of S_{n_1} and S_{n_2} are $(q+pz)^{n_1}$ and $(q+pz)^{n_2}$, respectively. Hence, $S_{n_1} + S_{n_2}$ has the generating function $(q+pz)^{n_1}(q+pz)^{n_2} = (q+pz)^{n_1+n_2}$, which is recognized as the generating function of a binomial distribution with parameters $n = n_1 + n_2$ and p.

Chapter 2, Exercise 8

'Verify the parenthetical statement of part c of Exercise 6'

$$P^n(z,t) = \left[\frac{z e^{-\lambda t}}{1 - z(1-e^{-\lambda t})}\right]^n = e^{-n\lambda t} z^n (1 - z(1-e^{-\lambda t}))^{-n}$$

$$= e^{-n\lambda t} z^n \Big(1 + n[z(1-e^{-\lambda t})] + \frac{n(n+1)}{2!}[z(1-e^{-\lambda t})]^2 + \cdots$$
$$\cdots + \frac{(n+r-1)!}{(n-1)!\,r!}[z(1-e^{-\lambda t})]^r + \cdots\Big)$$

$$= e^{-n\lambda t} z^n \sum_{r=0}^{\infty} \binom{n+r-1}{r} (1-e^{-\lambda t})^r z^r$$

$$= \sum_{j=n}^{\infty} \binom{j-1}{j-n} e^{-n\lambda t}(1-e^{-\lambda t})^{j-n} z^j.$$

Thus, $P_j(t) = \binom{j-1}{j-n} e^{-n\lambda t}(1-e^{-\lambda t})^{j-n}$ for $j \geq n$, and $P_j(t) = 0$ for $j < n$. $\square$

Chapter 2, Exercise 9

'Repeat Exercise 7, with the phrase...'

S_{n_ν} ($\nu = 1,2$) is the sum of n_ν independent, identically distributed random variables following a geometric distribution. Hence, $S_{n_1} + S_{n_2}$ is the sum of $n_1 + n_2$ independent, identically distributed random variables with a geometric distribution. Thus, $S_{n_1} + S_{n_2}$ has the negative binomial distribution (5.8) with $n = n_1 + n_2$.

Alternatively, the probability generating function (p.g.f.) of S_{n_1} equals $(p/(1-qz))^{n_1}$ and the p.g.f. of S_{n_2} equals $(p/(1-qz))^{n_2}$. Hence, $S_{n_1} + S_{n_2}$ has the p.g.f. $(p/(1-qz))^{n_1}(p/(1-qz))^{n_2} = (p/(1-qz))^{n_1+n_2}$, which is the p.g.f. of a variable with the negative binomial distribution (5.8) with $n = n_1 + n_2$.

Chapter 2, Exercise 10

'Feller [1971]. Find the distribution function of the length of...'

Let T $(0 \leqq T < c)$ be the length of the covering arc.

$$F(t) = P\{T \leqq t\} = \left(\frac{t}{c}\right)^2 \qquad (0 \leqq t \leqq c).$$

$$f(t) = \frac{dF(t)}{dt} = \frac{2t}{c^2} \qquad (0 \leqq t \leqq c).$$

$$E(T) = \int_0^c t f(t)\,dt = \frac{2}{3}c.$$

Chapter 2, Exercise 11

'Let $X_1, \ldots, X_n$ $(n \geqq 2)$ be...'

R is the maximum of the $n-1$ residual variables at $t = X_{(1)}$. $R \leqq x$ if and only if all of these $n-1$ exponential variables are less than or equal to x. Hence,

$$P\{R \leqq x\} = (1 - e^{-\mu x})^{n-1}.$$

□

Chapter 2, Exercise 12

'Let $X_1, \dots, X_n$ be a sequence of...'

Suppose S_n is the maximum of n independent exponential variables with mean μ^{-1}. Then $P\{S_n \le t\} = (1-e^{-\mu t})^n$.

Now, S_n may be decomposed into n successive time intervals of lengths $X_n, X_{n-1}, \dots, X_1$ such that $\{X_n\}$ is a set of independent exponential variables and X_j has mean $j\mu$. Since $\Sigma X_j = S_n$, $P\{\Sigma X_j \le t\} = P\{S_n \le t\} = (1-e^{-\mu t})^n$.

Chapter 2, Exercise 13

'In reliability theory the failure rate function $r(t)$...'

Suppose $F(t)$ is continuous and differentiable. The probability of a failure in $(t, t+\Delta t)$, given that no failure has occurred before t, equals

$$R(t,t+\Delta t) = \frac{F(t+\Delta t) - F(t)}{1 - F(t)},$$

whereby

$$r(t) \equiv \lim_{\Delta t \to 0} \frac{R(t,t+\Delta t)}{\Delta t} = \frac{f(t)}{1-F(t)}.$$

Thus, $r(t)dt$ has the desired interpretation. If $F(t) = 1 - e^{-\lambda t}$ $(t \ge 0)$, then clearly $r(t) = \lambda$.

Chapter 2, Exercise 14

'Let $X_1, X_2, \dots, X_n$ be independent exponential random variables...'

By an easy generalization of (5.21), $P\{\min(X_1, X_2, \dots, X_n) > x\} = e^{-(\Sigma \mu_j)x}$. Thus $Y_i = \min(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$ is exponentially distributed with parameter $\Sigma_{j \ne i} \mu_j$. Now write (5.23) as $P\{X_i = \min(X_1, X_2)\} = \mu_i/(\mu_1+\mu_2)$ $(i = 1, 2)$. Using the fact that X_i and Y_i are independent exponential variables, we find that

$$P\{X_i = \min(X_1, X_2, \dots, X_n)\} = P\{X_i = \min(X_i, Y_i)\} = \frac{\mu_i}{\mu_i + \sum_{j \ne i} \mu_j} = \frac{\mu_i}{\mu_1 + \mu_2 + \dots + \mu_n}.$$

□

Chapter 2, Exercise 15

'Let X_1 and X_2 be independent exponential variables.'

Direct proof

Clearly,

$$P\{t < X_i < t+dt,\ X_i = \min(X_1, X_2)\} = e^{-(\mu_1+\mu_2)t}\mu_i\,dt.$$

Hence,

$$P\{X_i > t,\ X_i = \min(X_1,X_2)\} = \int_t^\infty e^{-(\mu_1+\mu_2)x}\mu_i\,dx = \frac{\mu_i}{\mu_1+\mu_2}\,e^{-(\mu_1+\mu_2)t}.$$

For $t = 0$,

$$P\{X_i = \min(X_1,X_2)\} = \frac{\mu_i}{\mu_1+\mu_2} \qquad (5.23)$$

Thus,

$$P\{X_i > t \mid X_i = \min(X_1,X_2)\} = \frac{P\{X_i > t,\ X_i = \min(X_1,X_2)\}}{P\{X_i = \min(X_1,X_2)\}}$$

$$= e^{-(\mu_1+\mu_2)t}$$

$$= P\{\min(X_1,X_2) > t\}.$$

Proof by use of Markov property

The Markov property of the two exponential distributions implies

$$P\{X_i = \min(X_1,X_2) \mid \min(X_1,X_2) > t\} = P\{X_i = \min(X_1,X_2)\}.$$

By this and the formula $P\{A|B\} = P\{A\}P\{B|A\}/P\{B\}$,

$$P\{\min(X_1,X_2) > t \mid X_i = \min(X_1,X_2)\} = \frac{P\{\min(X_1,X_2)>t\}\ P\{X_i = \min(X_1,X_2) \mid \min(X_1,X_2)>t\}}{P\{X_i = \min(X_1,X_2)\}}$$

$$= P\{\min(X_1,X_2) > t\}$$

which is the same as

$$P\{X_i > t \mid X_i = \min(X_1,X_2)\} = P\{\min(X_1,X_2) > t\}$$

Generalization to $n \geq 2$ independent exponential variables is straightforward. $\square$

Chapter 2, Exercise 16

'At $t=0$ a customer (the test customer) places a request...'

[a] The departure rate from system, and thus from line into service, equals $s\mu$ as long as any customer is in the waiting line. Hence, $X_1, X_2, \ldots, X_{j+1}$ are independent exponential variables with mean $(s\mu)^{-1}$.

[b] $$E(X) = E\Big(\sum_{i=1}^{j+1} X_i\Big) = \sum_{i=1}^{j+1} E(X_i) = (j+1)(s\mu)^{-1}.$$

[c] $$E(T) = E(X) + \frac{1}{s\mu} + \frac{1}{(s-1)\mu} + \cdots + \frac{1}{\mu} = \frac{1}{\mu}\Big(\frac{j+1}{s} + \sum_{i=1}^{s} \frac{1}{i}\Big)$$

[d] $$P\{X=m\} = \begin{cases} 0 & (m = 1,2,\ldots,j+1), \\ \frac{1}{s} & (m = 1+j+1,\ 2+j+1, \ldots, s+j+1). \end{cases}$$

[e] $$P = \Big(1-\frac{1}{s}\Big)\frac{1}{2}.$$

Chapter 2, Exercise 17

'Suppose customers arrive at instants $T_1, T_2, \ldots$'

Clearly, $P_0(t) = 1 - G(t)$, and $P_j(t) = \int_0^t P_{j-1}(t-\xi)\,dG(\xi)$ for $j = 1, 2, \ldots$. Assume $G(x) = 1 - e^{-\lambda x}$. Then we have $P_0(t) = e^{-\lambda t}$, and

$$P_1(t) = \int_0^t P_0(t-\xi)\,dG(\xi) = \int_0^t e^{-\lambda(t-\xi)}\lambda e^{-\lambda\xi}\,d\xi = \lambda t e^{-\lambda t}.$$

It is seen that (5.25) holds for $j=0$ and $j=1$. Suppose it holds for $j=k$, so that $P_k(t) = [(\lambda t)^k/k!]\,e^{-\lambda t}$. Then

$$\begin{aligned} P_{k+1}(t) &= \int_0^t P_k(t-\xi)\,dG(\xi) = \int_0^t \frac{[\lambda(t-\xi)]^k}{k!} e^{-\lambda(t-\xi)}\lambda e^{-\lambda\xi}\,d\xi \\ &= \frac{\lambda^{k+1}}{k!} e^{-\lambda t} \int_0^t (t-\xi)^k\,d\xi = \frac{\lambda^{k+1}}{k!} e^{-\lambda t} \int_0^t y^k\,dy \\ &= \frac{(\lambda t)^{k+1}}{(k+1)!} e^{-\lambda t}. \end{aligned}$$

We conclude that if $G(x) = 1 - e^{-\lambda x}$, then $\{P_j(t)\}$ is the Poisson distribution with parameter λt. □

Chapter 2, Exercise 18

'Prove equation (5.37)' – cf. Ex. 3 of Chap. 5.

$t \leqq y$: The event $I_t > y$ can occur in two mutually exclusive ways: (1) No arrivals in $[0, y]$; (2) An arrival at $\tau \in [0, t]$ and no arrivals in $(\tau, \tau + y]$. Thus

$$P\{I_t > y\} = e^{-\lambda y} + \int_0^t \lambda e^{-\lambda y} d\tau = e^{-\lambda y} + \lambda t e^{-\lambda y}.$$

Hence,

$$P\{I_t \leqq y\} = 1 - e^{-\lambda y} - \lambda t e^{-\lambda y}. \qquad (t \leqq y)$$

$t > y$: The event $I_t > y$ can occur in three mutually exclusive ways: (1) No arrivals in $[0, t]$; (2) An arrival at $\tau \in [0, t-y)$ and no arrivals in $(\tau, t]$; (3) An arrival at $\tau \in [t-y, t]$ and no arrivals in $(\tau, \tau + y]$. Thus

$$\begin{aligned} P\{I_t > y\} &= e^{-\lambda t} + \int_0^{t-y} \lambda e^{-\lambda(t-\tau)} d\tau + \int_{t-y}^t \lambda e^{-\lambda y} d\tau \\ &= e^{-\lambda t} + [e^{-\lambda y} - e^{-\lambda t}] + \lambda y e^{-\lambda y} \\ &= e^{-\lambda y} + \lambda y e^{-\lambda y}. \end{aligned}$$

Hence,

$$P\{I_t \leqq y\} = 1 - e^{-\lambda y} - \lambda y e^{-\lambda y} \qquad (t > y)$$

The two equations may be combined into

$$P\{I_t \leqq y\} = 1 - e^{-\lambda y} - \lambda \min(y, t) e^{-\lambda y}. \qquad (5.37)$$

Chapter 2, Exercise 19

'Let $F(x, y)$ be the limiting joint distribution function ...'

The formula

$$F(x, y) \equiv \lim_{t \to \infty} P\{R_t \leqq x, I_t \leqq y\} = 1 - e^{-\lambda x} - \lambda x e^{-\lambda y} \qquad (0 \leqq x \leqq y)$$

may be derived from eq. (7.20) of Chapter 5.

(Chap. 2, Ex. 19)

$\boxed{a}$
$$\lim_{t\to\infty} P\{I_t \le y\} = \lim_{t\to\infty} P\{R_t \le y, I_t \le y\}$$
$$= 1 - e^{-\lambda y} - \lambda y e^{-\lambda y}.$$

$\boxed{b}$
$$\lim_{t\to\infty} P\{R_t > x\} = \lim_{t\to\infty} (1 - P\{R_t \le x\})$$
$$= 1 - \lim_{t\to\infty} P\{R_t \le x\}$$
$$= 1 - \lim_{t\to\infty} P\{R_t \le x, I_t \le \infty\}$$
$$= e^{-\lambda x}.$$

$\boxed{c}$ It may be shown that
$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \int_x^\infty \left(\int_{\xi+y}^\infty f(\xi,\eta)\, d\eta \right) d\xi$$
where $f(x,y) = dF(x,y)/dxdy$ is the density function. By differentiation we find $f(x,y) = \lambda^2 e^{-\lambda y}$. Hence,
$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \int_x^\infty \left(\int_{\xi+y}^\infty \lambda^2 e^{-\lambda\eta}\, d\eta \right) d\xi$$
$$= \int_x^\infty \lambda e^{-\lambda(\xi+y)}\, d\xi$$
$$= e^{-\lambda x} e^{-\lambda y}.$$

$\boxed{d}$
$$\lim_{t\to\infty} P\{A_t > y\} = \lim_{t\to\infty} P\{R_t > 0, A_t > y\} = e^{-\lambda 0} e^{-\lambda y} = e^{-\lambda y} \qquad (5.34)$$
$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = e^{-\lambda x} e^{-\lambda y} \qquad [\text{by (c)}]$$
$$= \lim_{t\to\infty} P\{R_t > x\} \lim_{t\to\infty} P\{A_t > y\} \qquad [\text{by (b) \& (5.34)}]$$
$$= \lim_{t\to\infty} (P\{R_t > x\} P\{A_t > y\}).$$

$\boxed{e}$ $f(x,y) = \lambda^2 e^{-\lambda y}$ is constant on the interval $0 \le x \le y$ for any given y. Thus R_t is uniformly distributed throughout the covering interval. □

Chapter 2, Exercise 20

'A bus shuttles back and forth..'

By Eq. (1.5), the probability that an arbitrary passenger is one of a bus load of j people equals $P\{Y=j\}=jP_j/a$, where $a=\sum jP_j$. Evidently, $\Pi_j = P\{Y=j+1\}$. It follows that, for any distribution $\{P_j\}$,

$$\Pi_j = \frac{(j+1)P_{j+1}}{a} \qquad (j=0,1,2,\dots).$$

The condition, $P_j = \Pi_j$ for all j, is therefore equivalent to

$$\begin{aligned} P_0 &= \Pi_0 = \tfrac{1}{a}P_1, \\ P_1 &= \Pi_1 = \tfrac{2}{a}P_2, \\ &\ \vdots \\ P_{j-1} &= \Pi_{j-1} = \tfrac{j}{a}P_j, \\ &\ \vdots \end{aligned}$$

or,

$$\begin{aligned} P_1 &= \tfrac{a}{1}P_0 \\ P_2 &= \tfrac{a^2}{2!}P_0 \\ &\ \vdots \\ P_j &= \tfrac{a^j}{j!}P_0 \\ &\ \vdots \end{aligned}$$

As $\sum_{j=0}^{\infty} P_j = 1$, we must have $P_0 = \left(\sum_{j=0}^{\infty}\frac{a^j}{j!}\right)^{-1} = e^{-a}$. The inference is

$$P_j = \Pi_j \ (j=0,1,2,\dots) \Leftrightarrow P_j = \frac{a^j}{j!}e^{-a} \quad (j=0,1,2,\dots).$$

Chapter 2, Exercise 21

'Suppose customers arrive according to a Poisson process.'

$\boxed{a}$ By the theorem of total probability, for $j=0,1,\dots,$

$$P\{M=j\} = \int_0^{\infty} P\{M=j \mid X=t\}\,dH(t) = \int_0^{\infty}\frac{(\lambda t)^j}{j!}e^{-\lambda t}\,dH(t).$$

(Chap. 2, Ex. 21 a)

We need the conditional means

$$E(M|X) = \lambda X,$$
$$E(M^2|X) = V(M|X) + E^2(M|X) = \lambda X + (\lambda X)^2.$$

Unconditioning, we derive

$$E(M) = E(E(M|X)) = E(\lambda X) = \lambda E(X) = \lambda\tau$$

$$E(M^2) = E(E(M^2|X)) = E(\lambda X + (\lambda X)^2) = \lambda E(X) + \lambda^2 E(X^2)$$
$$= \lambda\tau + \lambda^2(\sigma^2 + \tau^2),$$

$$V(M) = E(M^2) - E^2(M) = \lambda\tau + \lambda^2\sigma^2.$$

$\boxed{b}$

$$dP\{X \le t, M = j\} = \frac{(\lambda t)^j}{j!} e^{-\lambda t} dH(t),$$
$$dP\{X \le t | M = j\} = \frac{dP\{X \le t, M = j\}}{P\{M = j\}} = \frac{1}{P\{M=j\}} \frac{(\lambda t)^j}{j!} e^{-\lambda t} dH(t),$$

$$E(X|M=j) = \int_0^\infty t\, dP\{X \le t | M = j\}$$
$$= \frac{1}{P\{M=j\}} \int_0^\infty t \frac{(\lambda t)^j}{j!} e^{-\lambda t} dH(t)$$
$$= \frac{j+1}{\lambda} \frac{1}{P\{M=j\}} \int_0^\infty \frac{(\lambda t)^{j+1}}{(j+1)!} e^{-\lambda t} dH(t).$$

By part (a) then,

$$E(X|M=j) = \frac{j+1}{\lambda} \frac{P\{M=j+1\}}{P\{M=j\}} \qquad (j = 0, 1, \ldots).$$

$\boxed{c}$ "If" — We assume that $P\{M=j\} = \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau}$ $(j = 0, 1, \ldots)$.

By part (b),
$$E(X|M=j) = \frac{j+1}{\lambda} \frac{P\{M=j+1\}}{P\{M=j\}}$$
$$= \frac{j+1}{\lambda} \left[\frac{(\lambda\tau)^{j+1}}{(j+1)!} e^{-\lambda\tau}\right] \Big/ \left[\frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau}\right]$$
$$= \tau \equiv E(X).$$

(Chap. 2, Ex. 21 c)

"Only if" – We assume that $E(X|M=j)=E(X)\equiv\tau$ $(j=0,1,\dots)$

By part (b),

$$\tau=\frac{1}{\lambda}\frac{P\{M=1\}}{P\{M=0\}}=\frac{2}{\lambda}\frac{P\{M=2\}}{P\{M=1\}}=\cdots=\frac{j}{\lambda}\frac{P\{M=j\}}{P\{M=j-1\}}\cdots.$$

Hence, $P\{M=j\}=\frac{\lambda\tau}{j}P\{M=j-1\}=\frac{(\lambda\tau)^j}{j!}P\{M=0\}$ for $j\geq 1$. Utilizing $\sum_{j=0}^{\infty}P\{M=j\}=1$ we find $P\{M=0\}=e^{-\lambda\tau}$. Thus,

$$P\{M=j\}=\frac{(\lambda\tau)^j}{j!}e^{-\lambda\tau}\quad(j=0,1,\dots).$$

$\boxed{d}$ Let t meet the condition $0<H(t)<1$. Then

$$P\{X\leq t|M=j\}=\frac{P\{X\leq t,M=j\}}{P\{M=j\}}=\frac{\int_0^t\frac{(\lambda x)^j}{j!}e^{-\lambda x}dH(x)}{\int_0^{\infty}\frac{(\lambda x)^j}{j!}e^{-\lambda x}dH(x)}=\frac{\int_0^t x^j e^{-\lambda x}dH(x)}{\int_0^{\infty}x^j e^{-\lambda x}dH(x)}.$$

Now define

$$A_j=\int_0^t x^j e^{-\lambda x}dH(x);\quad B_j=\int_t^{\infty}x^j e^{-\lambda x}dH(x).$$

Thus,

$$P\{X\leq t|M=j\}=\frac{A_j}{A_j+B_j}\quad(j=0,1,\dots).$$

Clearly,

$$A_{j+1}=\int_0^t x^{j+1}e^{-\lambda x}dH(x)\leq\int_0^t t\,x^j e^{-\lambda x}dH(x)=tA_j,$$

$$B_{j+1}=\int_t^{\infty}x^{j+1}e^{-\lambda x}dH(x)>\int_t^{\infty}t\,x^j e^{-\lambda x}dH(x)=tB_j.$$

Hence, $A_{j+1}/A_j\leq t<B_{j+1}/B_j$, whereby $A_{j+1}/A_j<B_{j+1}/B_j$. Thus, $B_{j+1}>(A_{j+1}/A_j)B_j$, so that

$$P\{X\leq t|M=j+1\}=\frac{A_{j+1}}{A_{j+1}+B_{j+1}}<\frac{A_{j+1}}{A_{j+1}+(A_{j+1}/A_j)B_j}=\frac{A_j}{A_j+B_j}=P\{X\leq t|M=j\}.$$

This proves that for all t such that $0<H(t)<1$,

$$P\{X\leq t|M=j+1\}<P\{X\leq t|M=j\}\quad(j=0,1,\dots).\qquad(*)$$

The variable X, given $M=j+1$, stochastically dominates X, given $M=j$.

(Chap. 2, Ex. 21 d)

It remains to be shown that the stochastic dominance implies a higher mean. A simple proof is this:

$$\begin{aligned} E(X \mid M=j+1) &= \int_0^\infty [1-P\{X \le t \mid M=j+1\}]\,dt \\ &> \int_0^\infty [1-P\{X \le t \mid M=j\}]\,dt \qquad [\text{by } (*)] \\ &= E(X \mid M=j) \qquad (j=0,1,\ldots). \end{aligned}$$

We have used the fact that the mean of a nonnegative variable X with distribution function $F(t)=P\{X \le t\}$ is given by the formula $E(X) = \int_0^\infty t\,dF(t) = \int_0^\infty [1-F(t)]\,dt$, which may be proved by integration by parts.

$\boxed{e}$ $H(t) = 1-e^{-\mu t}$. Given exponential "service time", the process may be viewed as an exponential race, repeated until the μ-variable wins. By (5.23), the winning probabilities are $\lambda/(\lambda+\mu)$ and $\mu/(\lambda+\mu)$, respectively, for the λ- and μ-variable. We deduce that in this case

$$P\{M=j\} = \left(\frac{\lambda}{\lambda+\mu}\right)^j \frac{\mu}{\lambda+\mu} \qquad (j=0,1,\ldots).$$

$\boxed{f}$ $H(t) = 1-e^{-\mu t}$. Inserting the above expression for $P\{M=j\}$ into the formula in part (b) we easily derive $E(X \mid M=j) = (j+1)/(\lambda+\mu)$. Alternatively, if $M=j$, then the "service time" X is composed of $j+1$ intervals resulting from exponential races. By Ex. 15 these intervals are independent, exponential variables with mean $1/(\lambda+\mu)$. Proved!

Chapter 2, Exercise 22

'Customers request service from a group of s servers ...'

$\boxed{a}$ $$P = \frac{\lambda}{\lambda+s\mu},$$

$\boxed{b}$ $$P\{N=j\} = \left(\frac{\lambda}{\lambda+s\mu}\right)^j \frac{s\mu}{\lambda+s\mu},$$

$\boxed{c}$ $$P = \frac{s\mu}{\lambda+s\mu} \times \frac{(s-1)\mu}{\lambda+(s-1)\mu}.$$

□

Chapter 2, Exercise 23

'Continuation of Exercise 5'.

In Exercise 5 it was shown that

"If procedures (a) and (b) are equivalent, then $g(x)g(y) = g^2(\frac{x+y}{2})$" (*)

Using (*) we shall prove that

"Procedures (a) and (b) are equivalent if and only if N_ν ($\nu = 1,2$) has a Poisson distribution" (**)

"If": We assume that N_ν ($\nu = 1,2$) has a Poisson distribution with mean a. By procedure (a), the contents of the cells will be $J = N_1$ and $K = N_2$, respectively, and so J and K are independent Poisson variables with means a. By procedure (b), $N_1 + N_2$ is a Poisson variable with mean $2a$. The decomposition property, expressed by (5.45), implies that J and K will be independent Poisson variables with means $\frac{1}{2} \times 2a = a$. Hence, procedures (a) and (b) are equivalent.

"Only if": We assume that procedures (a) and (b) are equivalent. By (*),

$$g(x)g(y) = g^2(\tfrac{x+y}{2}) \tag{1}$$

Hence,

$$g(x)g(y) = g(x+y)\,g(0) \tag{2}$$

This implies $g(0) > 0$. Now put

$$g(z) = u(z)\,g(0) \tag{3}$$

Inserting (3) into (2) yields

$$u(x)\,u(y) = u(x+y) \tag{4}$$

Since $g(z)$ is increasing in z, so is $u(z)$, by (3). Eq. (4) is identical to Eq. (5.20). It follows that the only increasing function u satisfying the functional equation (4) is of the form

$$u(z) = e^{az} \quad (a>0).$$

(Chap. 2, Ex. 23)

Thus, by (3),

$$g(z) = e^{az} g(0) \tag{5}$$

But $g(1)=1$, so by (5) $1 = e^{a} g(0)$, whence $g(0) = e^{-a}$. Thus

$$g(z) = e^{-a(1-z)} \qquad (a>0). \tag{6}$$

This is recognized as the p.g.f. of a Poisson variable with mean a. Thus N_ν $(\nu = 1,2)$ has a Poisson distribution.

Chapter 2, Exercise 24

'Consider the single-server queue with an unlimited number..'

$\boxed{a}$ First multiply equation j $(j=0,1,\ldots)$ of equation system (1) by z^j:

$$\Pi_0^* z^0 = p_0 z^0 \Pi_0^* + (p_0 \Pi_1^*) z^0$$

$$\Pi_1^* z^1 = p_1 z^1 \Pi_0^* + (p_0 \Pi_2^* + p_1 \Pi_1^*) z^1$$

$$\Pi_2^* z^2 = p_2 z^2 \Pi_0^* + (p_0 \Pi_3^* + p_1 \Pi_2^* + p_2 \Pi_1^*) z^2$$

$$\vdots$$

Adding all these equations results in

$$\begin{aligned} g(z) &= h(z)\Pi_0^* + z^{-1}(p_0 + p_1 z + p_2 z^2 + \cdots)\sum_{i=1}^{\infty} \Pi_i^* z^i \\ &= h(z)\Pi_0^* + z^{-1} h(z)\left[g(z) - \Pi_0^*\right] \end{aligned}$$

Hence,

$$g(z) = \frac{(z-1)\,h(z)}{z - h(z)}\,\Pi_0^*. \tag{3}$$

$\boxed{b}$

$$\begin{aligned} h(z) &= \sum_{j=0}^{\infty} p_j z^j = \sum_{j=0}^{\infty}\left(\int_0^{\infty} \frac{(\lambda\xi)^j}{j!} e^{-\lambda\xi}\, dH(\xi)\right) z^j \\ &= \sum_{j=0}^{\infty} \int_0^{\infty} \frac{(\lambda z\xi)^j}{j!} e^{-\lambda\xi}\, dH(\xi) = \int_0^{\infty}\left(\sum_{j=0}^{\infty} \frac{(\lambda z\xi)^j}{j!}\right) e^{-\lambda\xi}\, dH(\xi) \\ &= \int_0^{\infty} e^{-(\lambda-\lambda z)\xi}\, dH(\xi) = \eta(\lambda - \lambda z). \end{aligned}$$

(Chap. 2, Ex. 24 b)

Substitution of $h(z) = \eta(\lambda - \lambda z)$ into (3) yields

$$g(z) = \frac{(z-1)\eta(\lambda-\lambda z)}{z - \eta(\lambda - \lambda z)} \Pi_0^* \tag{4}$$

$\boxed{c}$ By the application of l'Hospital's rule to Eq. (3),

$$g(1) = \lim_{z \to 1} g(z) = \frac{\frac{d}{dz}((z-1)h(z))\big|_{z=1}}{\frac{d}{dz}(z - h(z))\big|_{z=1}} \Pi_0^*$$

$$= \frac{(z-1)h'(z) + h(z)\big|_{z=1}}{1 - h'(z)\big|_{z=1}} \Pi_0^*$$

$$= \frac{h(1)}{1 - h'(1)} \Pi_0^* = \frac{\Pi_0^*}{1 - h'(1)}.$$

Clearly,

$$g(1) = 1. \tag{5}$$

Hence,

$$\Pi_0^* = 1 - h'(1).$$

Now, $h'(1) = \sum j p_j$ is the mean number of arrivals during a service time, which in Exercise 21a was shown to be equal to $\lambda\tau$. That is, $h'(1) = \lambda\tau$. For $\lambda\tau = \rho < 1$ then,

$$\Pi_0^* = 1 - \rho. \tag{6}$$

Thus Eq. (4) becomes

$$g(z) = \frac{(z-1)\eta(\lambda-\lambda z)}{z - \eta(\lambda-\lambda z)} (1-\rho) \qquad (\rho < 1). \tag{7}$$

$\boxed{d}$ By (4.5), $E(N^*) = g'(1)$. Differentiation of (3), with Π_0^* replaced by $1-\rho$, gives

$$g'(z) = \frac{A(z)}{B(z)} (1-\rho),$$

where

$$A(z) = h(z) - h^2(z) - z(1-z)h'(z),$$

and

$$B(z) = (z - h(z))^2.$$

(Chap. 2, Ex. 24 d)

Since $h(1)=1$, $A(1)=0$ and $B(1)=0$. Thus, evaluation of $g'(1)$ requires the application of L'Hospital's rule. Differentiation yields

$$A'(z) = 2z\,h'(z) - 2h(z)h'(z) - z(1-z)h''(z),$$

$$B'(z) = 2(z-h(z))(1-h'(z)).$$

As $A'(1)=0$ and $B'(1)=0$ we differentiate once more:

$$A''(z) = 2h'(z)(1-h'(z)) + (4z-1-2h(z))h''(z) - z(1-z)h'''(z),$$

$$B''(z) = 2(1-h'(z))^2 - 2(z-h(z))h''(z).$$

$A''(z)$ and $B''(z)$ have to be evaluated at $z=1$. We already know that $h(1)=1$ and $h'(1)=\rho$. In order to find $h''(1)$, recall that by part (b) $h(z)=\eta(\lambda-\lambda z)$. Hence $h'(z)=-\lambda\eta'(\lambda-\lambda z)$ and $h''(z)=\lambda^2\eta''(\lambda-\lambda z)$. Thus $h''(1)=\lambda^2\eta''(0)$. By definition, $\eta(s)=\int_0^\infty e^{-s\xi}dH(\xi)$. Hence, $\eta'(s)=\int_0^\infty(-\xi)e^{-s\xi}dH(\xi)$ and $\eta''(s)=\int_0^\infty\xi^2e^{-s\xi}dH(\xi)$. Thus $\eta''(0)=\int_0^\infty\xi^2dH(\xi)=\sigma^2+\tau^2$. It follows that $h''(1)=\lambda^2(\sigma^2+\tau^2)$. By these results,

$$A''(1) = 2\rho(1-\rho) + \lambda^2(\sigma^2+\tau^2),$$

$$B''(1) = 2(1-\rho)^2.$$

By l'Hospital's rule, $g'(1)=\frac{A''(1)}{B''(1)}(1-\rho)$. Hence

$$E(N^*) = \rho + \frac{\rho^2(1+(\sigma^2/\tau^2))}{2(1-\rho)}. \tag{8}$$

$\boxed{e}$ Let $H(\xi)=1-e^{-\mu\xi}$. Then $\eta(s)=\int_0^\infty e^{-s\xi}dH(\xi)=\int_0^\infty e^{-s\xi}\mu e^{-\mu\xi}d\xi=\mu/(\mu+s)$. Thus $\eta(\lambda-\lambda z)=\mu/(\mu+\lambda-\lambda z)$ and, by (7),

$$\begin{aligned} g(z) &= \frac{(z-1)\mu/(\mu+\lambda-\lambda z)}{z-\mu/(\mu+\lambda-\lambda z)}(1-\rho) = \frac{(z-1)\mu}{(z-1)(\mu-\lambda z)}(1-\rho) \\ &= (1-\rho)/(1-\rho z) = \sum_{j=0}^\infty(1-\rho)\rho^j z^j \qquad [\rho=\lambda\tau=\lambda/\mu] \end{aligned}$$

Hence,

$$\pi_j^* = (1-\rho)\rho^j \qquad (j=0,1,\ldots). \tag{9}$$

$\square$

Chapter 2, Exercise 25

'An operations research consultant ...'

No comment.

Chapter 2, Exercise 26

'In the model of Exercise 25, let X be the merging time ...'

In the model of Exercise 25 the (a priori) interarrival times $U_1, U_2, \ldots$ are i.i.d. exponential variables with mean α^{-1}, and the required acceleration times $V_1, V_2, \ldots$ are i.i.d. exponential variables with mean β^{-1}. As a consequence, the merging time $X = U_1 + U_2 + \cdots U_{n-1} + V_n$ is exponentially distributed with mean β^{-1} and does not depend on α (see preceding pages of book).

In the model of this exercise, it will be assumed that $V_1 = V_2 = \cdots = V$, where V is an exponential variable with mean β^{-1}. In this case the mean and the variance of X are a function of α. Both mean and variance are greater than in the model of Exercise 25 for identical α and β.

$\boxed{a}$ Let V equal the constant c. Then

$$X = c + \sum_{i=1}^{n-1} U_i, \qquad (1)$$

where each of the observed U_i's $(i=1,\ldots,n-1)$ has the conditional distribution of U, given $U \leq c$. Since these U_i's are i.i.d. variables and independent of $n-1$,

$$E(X|V=c) = c + E(n-1)E(U|U\leq c). \qquad (2)$$

Now, $n-1$ has the geometric distribution with parameter $p = \mathrm{Prob}\{U>c\} = e^{-\alpha c}$. Hence $E(n-1) = (1-p)/p = e^{\alpha c} - 1$. Substitution into (2) yields the desired expression

$$E(X|V=c) = c + (e^{\alpha c}-1)E(U|U\leq c). \qquad (3)$$

(Chap. 2, Ex. 26 b)

$\boxed{b}$ First we determine $E(U|U \le c)$. By assumption, U is exponentially distributed with mean α^{-1}, i.e. $\text{Prob}\{U \le u\} = 1 - e^{-\alpha u}$. Hence, $F_c(u) \equiv \text{Prob}\{U \le u \mid U \le c\} = (1-e^{-\alpha u})/(1-e^{-\alpha c})$, and the density function of U, given $U \le c$, is $f_c(u) = dF_c(u)/du = \alpha e^{-\alpha u}/(1-e^{-\alpha c})$, for $0 \le u \le c$. Consequently,

$$E(U|U \le c) = \int_0^c u f_c(u)du = \frac{1}{1-e^{-\alpha c}} \int_0^c u\alpha e^{-\alpha u} du = \frac{1}{1-e^{-\alpha c}}\left(-\frac{(1+\alpha u)e^{-\alpha u}}{\alpha}\Big|_0^c\right).$$

Hence,

$$E(U|U \le c) = \frac{1}{\alpha} - \frac{c}{e^{\alpha c}-1}. \tag{4}$$

Substitution of Equation (4) into Equation (3) results in

$$E(X|V=c) = \frac{e^{\alpha c}-1}{\alpha}. \tag{5}$$

We now assume that V is exponentially distributed with mean β^{-1}. Unconditioning on V we find

$$E(X) = \int_0^\infty E(X|V=c)\beta e^{-\beta c}dc = \frac{\beta}{\alpha}\left(\int_0^\infty e^{-(\beta-\alpha)c}dc - \int_0^\infty e^{-\beta c}dc\right),$$

by which

$$E(X) = \begin{cases} (\beta-\alpha)^{-1} & \text{when } \beta > \alpha, \\ \infty & \text{when } \beta \le \alpha. \end{cases} \tag{6}$$

$\boxed{c}$ For the purpose of calculating $V(X)$ we shall employ the decomposition formula

$$V(X) = E_c(V(X|V=c)) + V_c(E(X|V=c)), \tag{7}$$

which gives $V(X)$ as the sum of the mean of the conditional variance, given V, and the variance of the conditional mean, given V. Clearly,

$$V(X|V=c) = V\left(c+\sum_{i=1}^{n-1} U_i\right) = V\left(\sum_{i=1}^{n-1} U_i\right).$$

The distribution of $\sum_{i=1}^{n-1} U_i$ is a compound distribution. Since the formulas in part (b) of Exercise 4 also hold for nondiscrete variables (see Chap. 5, Ex. 5) we have

(Chap. 2, Ex. 26 c)

$$V(X|V=c) = E(n-1)V(U|U\leq c) + V(n-1)E^2(U|U\leq c) \qquad (8)$$

By part (a), $n-1$ is geometrically distributed with parameter $p=e^{-\alpha c}$. Hence, $E(n-1)=e^{\alpha c}-1$ and $V(n-1)=(1-p)/p^2=e^{\alpha c}(e^{\alpha c}-1)$. $E(U|U\leq c)$ is given by Eq. (4). Only $V(U|U\leq c)$ remains to be calculated. Following the development in part (b) we find

$$\begin{aligned} V(U|U\leq c) &= E(U^2|U\leq c) - E^2(U|U\leq c) \\ &= \int_0^c u^2 f_c(u)\,du - \left(\frac{1}{\alpha} - \frac{c}{e^{\alpha c}-1}\right)^2 \\ &= -\left.\frac{(\alpha^2u^2+2\alpha u+2)\,e^{-\alpha u}}{\alpha^2(1-e^{-\alpha c})}\right|_0^c - \left(\frac{1}{\alpha} - \frac{c}{e^{\alpha c}-1}\right)^2. \end{aligned}$$

Hence,

$$V(U|U\leq c) = \frac{1}{\alpha^2} - \frac{c^2e^{\alpha c}}{(e^{\alpha c}-1)^2}. \qquad (9)$$

Substitution of the various expressions into (8) gives

$$V(X|V=c) = (e^{\alpha c}-1)\left(\frac{1}{\alpha^2} - \frac{c^2e^{\alpha c}}{(e^{\alpha c}-1)^2}\right) + e^{\alpha c}(e^{\alpha c}-1)\left(\frac{1}{\alpha} - \frac{c}{e^{\alpha c}-1}\right)^2,$$

that reduces to

$$V(X|V=c) = \frac{1}{\alpha^2}\left(e^{2\alpha c} - 2\alpha c\, e^{\alpha c} - 1\right). \qquad (10)$$

Assuming that V is exponentially distributed with mean β^{-1} we have

$$\begin{aligned} E_c(V(X|V=c)) &= \int_0^\infty V(X|V=c)\,\beta e^{-\beta c}\,dc \\ &= \int_0^\infty \frac{e^{2\alpha c}-2\alpha c\,e^{\alpha c}-1}{\alpha^2}\,\beta e^{-\beta c}\,dc \\ &= \frac{\beta}{\alpha^2}\left(\int_0^\infty e^{-(\beta-2\alpha)c}\,dc - 2\alpha\int_0^\infty c\,e^{-(\beta-\alpha)c}\,dc - \int_0^\infty e^{-\beta c}\,dc\right) \end{aligned}$$

If $\beta\leq 2\alpha$, then $E_c(V(X|V=c))=\infty$. If, on the other hand, $\beta>2\alpha$, then

$$\begin{aligned} E_c(V(X|V=c)) &= \frac{\beta}{\alpha^2}\left(\frac{1}{\beta-2\alpha} - 2\alpha\left(-\left.\frac{1+(\beta-\alpha)c}{(\beta-\alpha)^2}\,e^{-(\beta-\alpha)c}\right|_0^\infty\right) - \frac{1}{\beta}\right) \\ &= \frac{\beta}{\alpha^2}\left(\frac{1}{\beta-2\alpha} - \frac{2\alpha}{(\beta-\alpha)^2} - \frac{1}{\beta}\right). \end{aligned}$$

(Chap 2, Ex. 26 c (cont'd))

Hence,

$$E_c(V(X|V=c)) = \begin{cases} \frac{2\alpha}{(\beta-2\alpha)(\beta-\alpha)^2} & \text{when } \beta > 2\alpha, \\ \infty & \text{when } \beta \leq 2\alpha. \end{cases} \tag{11}$$

Still under assumption of an exponentially distributed V with mean β^{-1}, we derive the second term on the right-hand side of (7):

$$\begin{aligned} V_c(E(X|V=c)) &= \int_0^\infty (E(X|V=c) - E(X))^2 \beta e^{-\beta c} dc \\ &= \int_0^\infty \left(\frac{e^{\alpha c}-1}{\alpha} - \frac{1}{\beta-\alpha}\right)^2 \beta e^{-\beta c} dc \qquad [\text{by (5) \& (6)}] \\ &= \frac{\beta}{\alpha^2}\int_0^\infty \left(e^{-(\beta-2\alpha)c} - 2e^{-(\beta-\alpha)c} + e^{-\beta c}\right) dc \\ &\quad - \frac{2\beta}{\alpha(\beta-\alpha)}\int_0^\infty \left(e^{-(\beta-\alpha)c} - e^{-\beta c}\right) dc \\ &\quad + \frac{\beta}{(\beta-\alpha)^2}\int_0^\infty e^{-\beta c} dc. \end{aligned}$$

If $\beta \leq 2\alpha$, then $V_c(E(X|V=c)) = \infty$. Otherwise

$$V_c(E(X|V=c)) = \frac{\beta}{\alpha^2}\left(\frac{1}{\beta-2\alpha} - \frac{2}{\beta-\alpha} + \frac{1}{\beta}\right) - \frac{2\beta}{\alpha(\beta-\alpha)}\left(\frac{1}{\beta-\alpha} - \frac{1}{\beta}\right) + \frac{1}{(\beta-\alpha)^2}.$$

Hence,

$$V_c(E(X|V=c)) = \begin{cases} \frac{\beta}{(\beta-2\alpha)(\beta-\alpha)^2} & \text{when } \beta > 2\alpha, \\ \infty & \text{when } \beta \leq 2\alpha. \end{cases} \tag{12}$$

By adding Equations (11) and (12) according to the decomposition formula, Eq. 7, we finally obtain, for the case of exponentially distributed characteristic gaps:

$$V(X) = \begin{cases} \frac{\beta+2\alpha}{\beta-2\alpha} \times \frac{1}{(\beta-\alpha)^2} & \text{when } \beta > 2\alpha, \\ \infty & \text{when } \beta \leq 2\alpha. \end{cases} \tag{13}$$

$\square$