Chapter 1, Exercise 1

'In what ways...'

No comment.

Chapter 1, Exercise 2

'List some applications of the Erlang loss model.'

No comment.

Chapter 1, Exercise 3

'Discuss ways...'

No comment.

Chapter 1, Exercise 4

'Extend the heuristic conservation-of-flow argument...'

In the present case only one-step state transitions are effected by arrivals or service completions. The conservation-of-flow principle then leads to the conclusion that in the long run there will be the same number of transitions $E_j \to E_{j+1}$ and $E_{j+1} \to E_j$, per unit time. Also, under certain conditions, the mean rate of transitions $E_j \to E_{j+1}$ is λP_j (true for Poisson arrivals), and the mean rate of transitions $E_{j+1} \to E_j$ is $(j+1)\tau^{-1} P_{j+1}$ for $j = 0, 1, \ldots, s-1$ and $s\tau^{-1} P_{j+1}$ for $j = s, s+1, \ldots$ (true for exponential service times). If these conditions hold, the conservation-of-flow equations extending equations (1.1) become

$$\lambda P_j = \begin{cases} (j+1)\tau^{-1} P_{j+1} & (j = 0, 1, \ldots, s-1), \\ s\tau^{-1} P_{j+1} & (j = s, s+1, \ldots). \end{cases} \qquad (*)$$

In the following eq. (*) are supposed to hold.

(Chap. 1, Ex. 4a)

[a] Recurrent solution of (*) results in

$$P_j = \begin{cases} \frac{a^j}{j!} P_0 & (j = 1, 2, \ldots, s-1), \\ \frac{a^j}{s!\, s^{j-s}} P_0 & (j = s, s+1, \ldots), \end{cases} \tag{1}$$

where $a = \lambda\tau$. Using (1) and $\sum_{j=0}^{\infty} P_j = 1$, we find

$$P_0 = \left(\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!\,(1 - a/s)} \right)^{-1}. \tag{2}$$

[b] In calculating (2) we set $\sum_{j=0}^{\infty} \left(\frac{a}{s}\right)^j = 1/(1 - a/s)$. However, this presupposes $a < s$. If $a \geq s$, then eq. (2) is incorrect and should be replaced by $P_0 = 0$.

The offered load $a = \lambda\tau$ equals the number of servers that on the average (in the long run) must be in service in order to dispose of the work load in such a way that customer orders do not pile up infinitely. Thus $a < s$ is necessary and sufficient for disposal of the work load without infinite delays.

[c] $$C(s,a) = \sum_{j=s}^{\infty} P_j = P_0 \sum_{j=s}^{\infty} \frac{a^j}{s!\, s^{j-s}} = P_0 \frac{a^s}{s!\,(1 - a/s)}.$$

By (2),

$$C(s,a) = \frac{\dfrac{a^s}{s!\,(1 - a/s)}}{\displaystyle\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!\,(1 - a/s)}}. \tag{3}$$

[d] In general, when $s > 1$, $C(s,a)$ will depend on the order in which waiting customers are selected from the queue. For example, if the customer with the shortest service time is always selected for service, then $C(s,a)$ will be different than when the converse policy is adopted. It is therefore necessary to specify service order. The usual assumption, making the model amenable to analysis, is that customers are selected without regard to service time required, as in

(Chap. 1, Ex. 4d)

order-of-arrival service. Without this assumption, the conservation-of-flow equations will not hold, even with Poisson arrivals and exponential service times.

[e] Clearly, $p_j = P_{s+j}/\sum_{k=0}^{\infty} P_{s+k} = P_{s+j}/C(s,a)$ for $j = 0,1,\ldots$. Hence, when $a<s$, by (1), (2) and (3),

$$p_j = (1-\tfrac{a}{s})(\tfrac{a}{s})^j = (1-\rho)\rho^j \quad (j=0,1,\ldots) \qquad (4)$$

where $\rho = a/s$.

[f]

$$\hat{p}_j = \frac{P_{s+k+j}}{\sum\limits_{i=0}^{\infty} P_{s+k+i}} = \frac{\frac{a^{s+k+j}}{s!\,s^{k+j}}P_0}{\sum\limits_{i=0}^{\infty}\frac{a^{s+k+i}}{s!\,s^{k+i}}P_0}$$

$$= \frac{\frac{a^j}{s^j}}{\sum\limits_{i=0}^{\infty}\frac{a^i}{s^i}} = (1-\tfrac{a}{s})(\tfrac{a}{s})^j.$$

Thus,

$$\hat{p}_j = (1-\rho)\rho^j = p_j, \qquad (j=0,1,\ldots)$$

independently of k.

[g] Assume $s=1$. By (1)

$$P_j = a^j P_0 \quad (j=0,1,\ldots),$$

where $P_0 = (1+a+a^2+\ldots)^{-1} = 1-a$, for $a<s=1$. Since $\rho = a$,

$$P_j = p_j = (1-\rho)\rho^j \quad (j=0,1,\ldots).$$

Finally, by (3),

$$C(1,a) = \frac{a/(1-a)}{1+a/(1-a)} = a.$$

□

Chapter 1, Exercise 5

'Consider the so-called loss-delay system...'

As in Exercise 4, we assume that the mean number of transitions $E_j \to E_{j+1}$ and $E_{j+1} \to E_j$ will be estimated correctly. That is the case with Poisson arrivals and exponential service times, respectively.

$\boxed{a}$ The conservation-of-flow argument leads to

$$\lambda P_j = \begin{cases} (j+1)\tau^{-1} P_{j+1} & (j = 0, 1, \ldots, s-1), \\ s\tau^{-1} P_{j+1} & (j = s, \ldots, s+n). \end{cases}$$

Hence,

$$P_j = \begin{cases} \frac{a^j}{j!} P_0 & (j = 1, 2, \ldots, s-1), \\ \frac{a^j}{s!\, s^{j-s}} P_0 & (j = s, \ldots, s+n), \end{cases}$$

where $a = \lambda\tau$, and, for all a, $P_0 = (\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!} \sum_{i=0}^{n} (\frac{a}{s})^i)^{-1}$.

$\boxed{b}$ $\underline{n = 0:}$

$$P_j = \frac{a^j}{j!} P_0 \quad (j = 0, 1, \ldots, s),$$

$$P_0 = (\sum_{k=0}^{s} \frac{a^k}{k!})^{-1}.$$

$$B(s,a) = P_s = \frac{a^s/s!}{\sum_{k=0}^{s} a^k/k!}. \qquad (1.6)$$

$\underline{n = \infty:}$

$$P_j = \begin{cases} \frac{a^j}{j!} P_0 & (j = 1, 2, \ldots, s-1), \\ \frac{a^j}{s!\, s^{j-s}} P_0 & (j = s, s+1, \ldots), \end{cases}$$

$$P_0 = (\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!(1-a/s)})^{-1}.$$

$$C(s,a) = \sum_{i=0}^{\infty} P_{s+i} = \frac{a^s/[s!(1-a/s)]}{\sum_{k=0}^{s-1} a^k/k! + a^s/[s!(1-a/s)]}.$$

$\boxed{c}$ When $n < \infty$, there is no restriction on a (the sums involved in the formula for P_0 are finite). When $n = \infty$, the restriction is $a < s$ as discussed in Exercise 4. □

Chapter 1, Exercise 6

'Consider a queueing model with two servers and one waiting position...'

a

$$\lambda P_0 = 1\tau^{-1} P_1,$$
$$\lambda P_1 = 2\tau^{-1} P_2,$$
$$p\lambda P_2 = 2\tau^{-1} P_3.$$

b

$$P_1 = a P_0, \qquad (a = \lambda\tau)$$
$$P_2 = \frac{a^2}{2} P_0,$$
$$P_3 = p\frac{a^3}{4} P_0,$$
$$P_0 = \left(1 + a + \frac{a^2}{2} + p\frac{a^3}{4}\right)^{-1}.$$

c

$$B = (1-p) P_2 + P_3$$

d Let $\lambda = 2$ and $\tau = 1$, so that $a = 2$, and let $p = 1/2$. Then, by part (b),

$$(P_0, P_1, P_2, P_3) = \left(\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{6}\right),$$

and by part (c),

$$B = \tfrac{1}{3}.$$

e

$$R = 2.00\,\lambda(P_0 + P_1) + 1.00\,p\lambda P_2$$
$$= 2\tfrac{1}{3}\ \$/h.$$

f

$$C = 0.50 \cdot 2 + 0.25[1 \cdot P_1 + 2(P_2 + P_3)]$$
$$= 1\tfrac{1}{3}\ \$/h.$$

Hence,

$$\text{Profit rate} = R - C = 2\tfrac{1}{3} - 1\tfrac{1}{3} = 1\ \$/h.$$

□