Chapter 5, Exercise 1

'(P. J. Burke [1968, unpublished].) Consider a birth-and-death process...'

$\boxed{a}$ Consider the Markov chain of states immediately following events, where an event is either an arrival (causing a change of state) or a departure. Denote by $E_{i_1}, E_{i_2}, E_{i_3}$, three such consecutive states in statistical equilibrium.

For each $j \geq 0$, by the Markov property,

$$\begin{aligned} P\{E_{i_2}=E_{j+1}, E_{i_3}=E_j\} &= P\{E_{i_1}=E_j, E_{i_2}=E_{j+1}, E_{i_3}=E_j\} \\ &+ P\{E_{i_1}=E_{j+2}, E_{i_2}=E_{j+1}, E_{i_3}=E_j\} \\ &= P\{E_{i_1}=E_j, E_{i_2}=E_{j+1}\} P\{E_{i_3}=E_j \mid E_{i_2}=E_{j+1}\} \\ &+ P\{E_{i_1}=E_{j+2}, E_{i_2}=E_{j+1}\} P\{E_{i_3}=E_j \mid E_{i_2}=E_{j+1}\}. \end{aligned}$$

Now,

$$\begin{aligned} P\{E_{i_2}=E_{j+1}, E_{i_3}=E_j\} &= \tfrac{1}{2}\pi_j^*, \\ P\{E_{i_1}=E_j, E_{i_2}=E_{j+1}\} &= \tfrac{1}{2}\pi_j, \\ P\{E_{i_1}=E_{j+2}, E_{i_2}=E_{j+1}\} &= \tfrac{1}{2}\pi_{j+1}^*. \end{aligned}$$

The first equation, for instance, holds because with probability 1/2 an event is a departure, and the conditional probability of departure state E_j, given a departure, equals π_j^*. Inserting these expressions and writing $P\{E_{i_3}=E_j \mid E_{i_2}=E_{j+1}\} = P\{E_{j+1} \to E_j\}$ we obtain, for each $j \geq 0$,

$$\pi_j^* = \pi_j P\{E_{j+1} \to E_j\} + \pi_{j+1}^* P\{E_{j+1} \to E_j\}. \qquad (*)$$

Clearly,

$$P\{E_{j+1} \to E_j\} = \frac{\mu_{j+1}}{\lambda_{j+1} + \mu_{j+1}},$$

and, by (3.3),

$$\pi_j^* = \pi_j$$

for $j = 0, 1, \ldots$. By substitution of these expressions into $(*)$, and reduction, we find

$$\lambda_{j+1}\pi_j = \mu_{j+1}\pi_{j+1} \quad (j = 0, 1, \ldots). \qquad (1)$$

$\boxed{b}$ For a birth-and-death process with n sources, suppose the arrival rate in state j (E_j), $j = 0, 1, \ldots, n-1$, depends on only the difference $n-j$; i.e. $\lambda_j[n] = f(n-j)$, where $f(\cdot) > 0$ is any function. By (1), $\lambda_{j+1}[n]\Pi_j[n] = \mu_{j+1}\Pi_{j+1}[n]$ $(j = 0, 1, \ldots, n-2)$, and as $\lambda_{j+1}[n] = f(n-j-1) = \lambda_j[n-1]$,

$$\lambda_j[n-1]\,\Pi_j[n] = \mu_{j+1}\Pi_{j+1}[n] \qquad (j = 0, 1, \ldots, n-2). \tag{2}$$

By Eq. (3.15) of Chapter 2, the outside observer's distribution in a system with $n-1$ sources will satisfy

$$\lambda_j[n-1]\,P_j[n-1] = \mu_{j+1}P_{j+1}[n-1] \quad (j = 0, 1, \ldots, n-2). \tag{3}$$

A comparison of (2) and (3) leads to the conclusion that

$$\Pi_j[n] = P_j[n-1] \qquad (j = 0, 1, \ldots, n-1) \tag{4}$$

for any finite-source birth-and-death process with $\lambda_j[n] = f(n-j)$ and $\mu_j > 0$ for $j > 0$.

Chapter 5, Exercise 2

'Burke's theorem'.

"For the M/M/s queue in equilibrium, the sequence of service completion epochs follows a Poisson process (with the same parameter as the input process); that is, the output process is statistically the same as the input process."

Let T_1 and T_2 be two arbitrary consecutive service completion epochs. Define $F_j(t)$ as the probability that simultaneously $T_2 > T_1 + t$ and the number of customers in system at $T_1 + t$ equals j. Let $\mu(j) = j\mu$ if $j \leq s$, $\mu(j) = s\mu$ if $j > s$.

$\boxed{a}$

$$F_0(t+h) = F_0(t)[1-\lambda h] + o(h),$$

$$F_j(t+h) = F_j(t)[1-(\lambda+\mu(j))h] + F_{j-1}(t)\lambda h + o(h) \qquad (j = 1, 2, \ldots).$$

(Chap. 5, Ex. 2 a)

Hence,

$$\frac{dF_0(t)}{dt} = -\lambda F_0(t),$$

$$\frac{dF_j(t)}{dt} = -(\lambda+\mu(j))F_j(t) + \lambda F_{j-1}(t) \qquad (j=1,2,\ldots).$$

with initial condition $F_j(0) = \pi_j^*$ for $j=0,1,2,\ldots$.

It is easily found that $F_0(t) = \pi_0^* e^{-\lambda t}$. We shall verify that the complete solution is

$$F_j(t) = \pi_j^* e^{-\lambda t} \qquad (t \geqq 0,\ j=0,1,2,\ldots). \qquad (1)$$

(1) has been shown to produce the right answer for $j=0$, and it clearly satisfies the initial condition for $j=1,2,\ldots$. Thus it remains to demonstrate that the equation satisfies the differential-difference equations above for $j=1,2,\ldots$. Substitution of (1) into the appropriate differential-difference equation and some simple calculation and reduction yield

$$\lambda \pi_{j-1}^* = \mu(j)\pi_j^* \qquad (j=1,2,\ldots). \qquad (2)$$

That this equation holds can be seen by making the substitution $\pi_j^* = \pi_j$, which results in a special case of Eq. (1) of Ex. (1), or making the substitution $\pi_j^* = P_j$, which results in the conservation-of-flow equation $\lambda P_{j-1} = \mu(j)P_j$, $j=1,2,\ldots$. We can therefore conclude that our Equation (1) indeed gives the desired probability $F_j(t)$, for all t and j.

$\boxed{b}$ Let $F(t)$ denote the probability that $T_2 > T_1 + t$; that is, $F(t) = P\{T_2 - T_1 > t\}$. By (1), and the definition of $F_j(t)$,

$$\begin{aligned} F(t) &= \sum_{j=0}^{\infty} F_j(t) \\ &= \sum_{j=0}^{\infty} \pi_j^* e^{-\lambda t} \\ &= e^{-\lambda t}. \end{aligned} \qquad (3)$$

Thus, the time separating two successive departures is exponentially distributed with the same mean as the interarrival times.

(Chap. 5, Ex. 2c)

c Let $x = T_2 - T_1$ and let $\hat{j}_2$ be the number of customers left behind by the departure at T_2. In equilibrium,

$$\begin{aligned} P\{\hat{j}_2 = j,\, x > t\} &= \int_t^\infty F_{j+1}(x)\,\mu(j+1)\,dx \\ &= \mu(j+1)\,\pi_{j+1}^* \int_t^\infty e^{-\lambda x}\,dx \qquad [\text{by (1)}] \\ &= \frac{\mu(j+1)}{\lambda}\,\pi_{j+1}^*\, e^{-\lambda t} \\ &= \pi_j^*\, e^{-\lambda t}. \qquad [\text{by (2)}] \end{aligned}$$

Thus,

$$P\{\hat{j}_2 = j,\, x > t\} = P\{\hat{j}_2 = j\}\,P\{x > t\}, \tag{4}$$

which means that the length of the interdeparture interval x and the number of customers in the system at the start of the next interval are independent variables.

Denote by T_3 the departure epoch subsequent to T_2. By the Markov property

$$P\{T_2 - T_1 > t \mid \hat{j}_2 = j,\, T_3 - T_2 = s\} = P\{T_2 - T_1 > t \mid \hat{j}_2 = j\}. \tag{5}$$

By (5) $T_2 - T_1$ may depend on $T_3 - T_2$ only through $\hat{j}_2$. But, by (4), $T_2 - T_1$ is independent of $\hat{j}_2$. Hence, $T_2 - T_1$ is independent of $T_3 - T_2$. An extension of this argument leads to the conclusion that all the interdeparture interval lengths are independent variables

Remark. Nowhere has the particular form of the function $\mu(j)$ been used. It is worth noting that the whole line of proof applies to any birth-and-death process with birth rate $\lambda_j = \lambda$ for all $j \geq 0$ and death rates $\mu_0 = 0$ and $\mu_j > 0$ for $j \geq 1$, where the λ and μ_j's only meet the condition for the existence of an equilibrium distribution. That is, the output process is a Poisson process with rate λ also in the general case, not just for an M/M/s queue.

□

Chapter 5, Exercise 3

'Solve Exercise 18 of Chapter 2 by evaluating the integral...'

$$F_{I_t}(y) = \int F_{R_t}(y-x)\, dF_{A_t}(x).$$

It is understood that $F_{R_t}(z) = P\{R_t \le z\}$, $F_{A_t}(z) = P\{A_t \le z\}$ and $F_{I_t}(z) = P\{I_t \le z\}$. The random variables R_t and A_t are forward and backward recurrence time at t, respectively; R_t and A_t are independent variables; and $I_t = R_t + A_t$. Recall that

$$F_{R_t}(z) = 1 - e^{-\lambda z} \quad (z \ge 0), \qquad [\text{by Eq. (5.30), Chap. 2}]$$

$$F_{A_t}(z) = \begin{cases} 1 - e^{-\lambda z} & (0 \le z < t), \\ 1 & (z \ge t). \end{cases} \qquad [\text{by Eq. (5.33), Chap. 2}]$$

Hence,
$$F_{I_t}(y) = \int_{x=0}^{y} F_{R_t}(y-x)\, dF_{A_t}(x) = \int_{x=0}^{y} [1 - e^{-\lambda(y-x)}]\, dF_{A_t}(x).$$

$\underline{y < t}$:
$$\begin{aligned} F_{I_t}(y) &= \int_0^y [1 - e^{-\lambda(y-x)}]\, \lambda e^{-\lambda x} dx \\ &= 1 - e^{-\lambda y} - \lambda y e^{-\lambda y}. \end{aligned}$$

$\underline{y \ge t}$:
$$\begin{aligned} F_{I_t}(y) &= \int_0^t [1 - e^{-\lambda(y-x)}]\, \lambda e^{-\lambda x} dx + [1 - e^{-\lambda(y-t)}]\, e^{-\lambda t} \\ &= 1 - e^{-\lambda y} - \lambda t e^{-\lambda y}. \end{aligned}$$

Thus, for all y,
$$F_{I_t}(y) = 1 - e^{-\lambda y} - \lambda \min(y,t) e^{-\lambda y}.$$

Chapter 5, Exercise 4

'Let N be a nonnegative, integer-valued random variable ...'

$$\begin{aligned} \phi(s) &= \int_0^\infty e^{-st} dF(t) = \sum_{j=0}^{\infty} e^{-sj} P\{N = j\} \\ &= \sum_{j=0}^{\infty} P\{N=j\} (e^{-s})^j \\ &= g(e^{-s}). \end{aligned}$$

□

Chapter 5, Exercise 5

'Consider again the premise of Exercise 4 of Chapter 2, but...'

X_i, $i=1,2,\ldots$, is a sequence of independent, identically distributed nonnegative random variables with distribution function $F(t) = P\{X \le t\}$ and Laplace-Stieltjes transform $\phi(s) = \int_0^\infty e^{-st} dF(t)$. N is a nonnegative, integer-valued random variable with generating function $g(z) = \sum_{n=0}^\infty P\{N=n\} z^n$. $\{X_i\}$ and N are independent.

[a] Let $S_0 = 0$ if $N=0$, and $S_N = \sum_{i=1}^N X_i$ if $N \ge 1$. Clearly,

$$P\{S_N \le t\} = \sum_{n=0}^\infty P\{N=n\} P\{S_n \le t\} = \sum_{n=0}^\infty P\{N=n\} F_n(t),$$

where $F_0(t) \equiv 1$, and $F_n(t)$, $n \ge 1$, is the n-fold convolution of $F(t)$. Thus, letting $\psi(s)$ denote the Laplace-Stieltjes transform of S_N,

$$\begin{aligned}\psi(s) &= \int_0^\infty e^{-st} dP\{S_N \le t\} \\ &= \sum_{n=0}^\infty P\{N=n\} \int_0^\infty e^{-st} dF_n(t) \\ &= \sum_{n=0}^\infty P\{N=n\} [\phi(s)]^n \qquad [\text{by } (6.5)] \\ &= g(\phi(s)).\end{aligned}$$

[b] By differentiating $\psi(s)$ twice we obtain

$$\psi'(s) = g'(\phi(s))\, \phi'(s),$$

$$\psi''(s) = g'(\phi(s))\, \phi''(s) + g''(\phi(s)) [\phi'(s)]^2.$$

As $\phi(0) = 1$,

$$\psi'(0) = g'(1)\, \phi'(0),$$

$$\psi''(0) = g'(1)\, \phi''(0) + g''(1) [\phi'(0)]^2.$$

Now, $g'(1) = E(N)$, $g''(1) = E(N^2) - E(N)$, $\phi'(0) = -E(X)$, $\phi''(0) = E(X^2)$. Hence,

$$\psi'(0) = -E(N)E(X), \qquad (1)$$

and $\psi''(0) = E(N)E(X^2) + [E(N^2) - E(N)] E^2(X)$, whereby

$$\psi''(0) = E(N)\, V(X) + E(N^2) E^2(X). \qquad (2)$$

(Chap. 5, Ex. 5)

Mean and variance of S_N may be derived from the Laplace-Stieltjes transform as follows

$$E(S_N) = -\psi'(0),$$

$$V(S_N) = E(S_N^2) - E^2(S_N) = \psi''(0) - [\psi'(0)]^2.$$

By (1) and (2), then,

$$E(S_N) = E(N)\,E(X),$$

$$V(S_N) = E(N)\,V(X) + V(N)\,E^2(X).$$

Chapter 5, Exercise 6

' We shall show in Section 5.8 that... '

" $W(t)$ in the M/G/1 queue with service in order of arrival has Laplace - Stieltjes transform $\omega(s) = \int_0^\infty e^{-st} dW(t)$ given by

$$\omega(s) = \frac{s(1-\rho)}{s - \lambda[1-\eta(s)]}, \qquad (1)$$

where $\eta(s) = \int_0^\infty e^{-st} dH(t)$ is the Laplace - Stieltjes transform of the service-time distribution function $H(t)$, with mean $\tau = \int_0^\infty t\,dH(t)$, and $\rho = \lambda\tau < 1$, where λ is the arrival rate."

$\boxed{a}$ We shall determine the mean wait from the relation $E(W) = -\omega'(0)$. Differentiation of (1) results in

$$\omega'(s) = -\frac{f(s)}{g(s)},$$

where

$$f(s) = \lambda(1-\rho)[1-\eta(s) + s\eta'(s)],$$

$$g(s) = (s - \lambda[1-\eta(s)])^2.$$

Observe that $f(0) = 0$ and $g(0) = 0$. However, $\omega'(0) = \lim_{s\to 0} \omega'(s)$ can be evaluated by a double application of l'Hospitals rule.

(Chap. 5, Ex. 6a)

First we calculate

$$f'(s) = \lambda(1-\rho)s\,\eta''(s),$$
$$g'(s) = 2(1+\lambda\eta'(s))(s-\lambda[1-\eta(s)]),$$

and, since $f'(0)=0$ and $g'(0)=0$, we differentiate again, obtaining

$$f''(s) = \lambda(1-\rho)\eta''(s) + \lambda(1-\rho)s\,\eta'''(s),$$
$$g''(s) = 2(1+\lambda\eta'(s))^2 + 2\lambda\eta''(s)(s-\lambda[1-\eta(s)]).$$

Hence,

$$f''(0) = \lambda(1-\rho)(\tau^2+\sigma^2),$$
$$g''(0) = 2(1-\rho)^2,$$

where σ^2 is the service-time variance, and we have used that $\lambda\eta'(0) = \lambda(-\tau) = -\rho$. Finally, from $E(W) = -\omega'(0) = f(0)/g(0) = f''(0)/g''(0)$,

$$E(W) = \frac{\rho\tau}{2(1-\rho)}\left(1+\frac{\sigma^2}{\tau^2}\right). \qquad (2)$$

$\boxed{b}$ When service-times are exponentially distributed with mean μ^{-1}, then the waiting-time distribution function $W(t)$ is

$$W(t) = \begin{cases} 0 & (t<0), \\ 1-\rho e^{-(1-\rho)\mu t} & (t \geq 0). \end{cases} \qquad (3)$$

Thus,

$$\begin{aligned}\omega(s) &= \int_{0-}^{\infty} e^{-st}\,dW(t) \\ &= e^{-s0}P\{W=0\} + \int_{0+}^{\infty} e^{-st}\,dW(t) \\ &= (1-\rho) + (1-\rho)\lambda\int_0^{\infty} e^{-(s+\mu-\lambda)t}\,dt \\ &= (1-\rho)\frac{s+\mu}{s+\mu-\lambda}.\end{aligned}$$

This result is in agreement with Equation (1), since in the case of an exponential service time distribution, we have $\eta(s) = \mu/(\mu+s)$, so that (1) becomes

$$\omega(s) = \frac{s(1-\rho)}{s-\lambda[1-\mu/(\mu+s)]} = (1-\rho)\frac{s+\mu}{s+\mu-\lambda}.$$

$\square$

Chapter 5, Exercise 7

'Show that if $G(\xi) = 1 - e^{-\lambda\xi}$ in (7.9), then $F(x) = 1 - e^{-\lambda x}$;...'

Since the interevent times have the exponential distribution with parameter λ, $G(\xi) = 1 - e^{-\lambda\xi}$, the mean of the interevent interval is

$$\beta = \int_0^\infty x\, dG(x) = \lambda^{-1}.$$

By (7.9) the equilibrium forward recurrence time has the distribution

$$\begin{aligned} F(x) &= \frac{1}{\beta}\int_0^x [1 - G(\xi)]\, d\xi \\ &= \lambda \int_0^x e^{-\lambda\xi}\, d\xi \\ &= 1 - e^{-\lambda x}, \end{aligned}$$

which is the same as the distribution of the interevent times.

Chapter 5, Exercise 8

'Verify Equation (7.13).'

The equilibrium forward recurrence distribution $F(x)$ is given by (7.9) and has the Laplace-Stieltjes transform

$$\phi(s) = \frac{1}{\beta}\,\frac{1-\gamma(s)}{s}. \qquad (7.10)$$

Hence,

$$\phi'(s) = -\frac{1}{\beta}\,\frac{s\gamma'(s) + 1 - \gamma(s)}{s^2}.$$

Since both numerator and denominator equal zero for $s = 0$, $\phi'(0)$ is evaluated by L'Hospital's rule.

$$\phi'(0) = -\frac{1}{\beta}\lim_{s\to 0}\frac{s\gamma''(s) + \gamma'(s) - \gamma'(s)}{2s} = -\frac{1}{\beta}\,\frac{\gamma''(0)}{2} = -\frac{1}{\beta}\,\frac{\beta^2+\sigma^2}{2}.$$

As $\beta^* = \int_0^\infty x\, dF(x)$ is determined by $\beta^* = -\phi'(0)$, we have

$$\beta^* = \frac{\beta}{2} + \frac{\sigma^2}{2\beta}. \qquad (7.13)$$

□

Chapter 5, Exercise 9

'a. Show that...'

$\boxed{a}$ Define $R_t = T_{j+1} - t$; $I_t = T_{j+1} - T_j$, where $T_j \le t < T_{j+1}$ for some j. Assume $0 \le x \le y < t$.
For given j ($j = 1, 2, \ldots$) and t we have

$$P\{T_j \le t < T_{j+1}, R_t \le x, I_t \le y\} = \int_{t-y}^{t-y+x} [G(y) - G(t-\xi)]\, dP\{T_j \le \xi\}$$

$$+ \int_{(t-y+x)+}^{t} [G(t-\xi+x) - G(t-\xi)]\, dP\{T_j \le \xi\}$$

The formula is a simple consequence of the following observations: (i) If $T_j \le t-y$ or $T_j > t$, then the event $\{T_j \le t < T_{j+1}, R_t \le x, I_t \le y\}$ cannot occur; (ii) If $t-y < T_j \le t-y+x$, then the event will occur if and only if $t - T_j < I_t \le y$; (iii) If $t-y+x < T_j \le t$, then the event will occur if and only if $t - T_j < I_t \le t - T_j + x$. For $j = 0$ we have $T_0 = 0 < t-y$, so that, by (i), the probability of the event is zero.
Obviously, $P\{R_t \le x, I_t \le y\} = \sum_{j=0}^{\infty} P\{T_j \le t < T_{j+1}, R_t \le x, I_t \le y\}$.
Thus

$$P\{R_t \le x, I_t \le y\} = \sum_{j=1}^{\infty} \int_{t-y}^{t-y+x} [G(y) - G(t-\xi)]\, dP\{T_j \le \xi\}$$

$$+ \sum_{j=1}^{\infty} \int_{(t-y+x)+}^{t} [G(t-\xi+x) - G(t-\xi)]\, dP\{T_j \le \xi\}$$

$$(0 \le x \le y < t)$$

$\boxed{b}$ Since $m(\xi) = \sum_{j=1}^{\infty} P\{T_j \le \xi\}$,

$$P\{R_t \le x, I_t \le y\} = \int_{t-y}^{t-y+x} [G(y) - G(t-\xi)]\, dm(\xi) + \int_{(t-y+x)+}^{t} [G(t-\xi+x) - G(t-\xi)]\, dm(\xi)$$

Letting $t \to \infty$ and using $\lim_{\xi \to \infty} \frac{dm(\xi)}{d\xi} = \frac{1}{\beta}$, we obtain

$$\lim_{t\to\infty} P\{R_t \le x, I_t \le y\} = \frac{1}{\beta} \int_{t-y}^{t-y+x} [G(y) - G(t-\xi)]\, d\xi$$

$$+ \frac{1}{\beta} \int_{(t-y+x)+}^{t} [G(t-\xi+x) - G(t-\xi)]\, d\xi .$$

(Chap. 5, Ex. 9 b)

By the substitution $t-\xi \to \xi$,

$$\lim_{t\to\infty} P\{R_t \leq x, I_t \leq y\} = \frac{1}{\beta}\int_{y-x}^{y} [G(y)-G(\xi)]\,d\xi + \frac{1}{\beta}\int_{0}^{y-x} [G(\xi+x)-G(\xi)]\,d\xi,$$

which by further rewriting becomes

$$\lim_{t\to\infty} P\{R_t \leq x, I_t \leq y\} = \frac{1}{\beta}\int_0^x G(y)\,d\xi - \frac{1}{\beta}\int_{y-x}^{y} G(\xi)\,d\xi$$
$$+ \frac{1}{\beta}\int_x^y G(\xi)\,d\xi - \frac{1}{\beta}\int_0^{y-x} G(\xi)\,d\xi.$$

Hence,

$$\lim_{t\to\infty} P\{R_t \leq x, I_t \leq y\} = \frac{1}{\beta}\int_0^x [G(y)-G(\xi)]\,d\xi \quad (0 \leq x \leq y). \qquad (7.20)$$

$\boxed{c}$ By setting $x = y$ in (7.20) we find

$$\lim_{t\to\infty} P\{I_t \leq y\} = \frac{1}{\beta}\int_0^y [G(y)-G(\xi)]\,d\xi$$
$$= \frac{1}{\beta}[G(y)-G(\xi)]\,\xi\Big|_0^y$$
$$- \frac{1}{\beta}\int_0^y \xi\,d(G(y)-G(\xi))$$

whereby

$$\lim_{t\to\infty} P\{I_t \leq y\} = \frac{1}{\beta}\int_0^y \xi\,dG(\xi).$$

By differentiation with respect to y and the subsequent substitution $y = x$, we derive

$$\lim_{t\to\infty} dP\{I_t \leq x\} = \frac{1}{\beta}\,x\,dG(x) \qquad (7.19)$$

which gives the equilibrium probability density of the covering interval.

It is also worth noting that by setting $y = \infty$ in (7.20) we find again

$$F(x) = \lim_{t\to\infty} P\{R_t \leq x\} = \frac{1}{\beta}\int_0^x [1-G(\xi)]\,d\xi \qquad (7.9)$$

<u>(Chap. 5, Ex. 9 d)</u>

$\boxed{d}$ We shall prove

$$\lim_{t\to\infty} P\{R_t \leqq x \mid I_t = y\} = \frac{x}{y} \qquad (0 \leqq x \leqq y). \qquad (7.21)$$

First (7.21) is proven under the assumption that $G(\xi)$ has a discontinuity point at $\xi = y$.

Obviously, in this case,

$$\lim_{t\to\infty} P\{R_t \leqq x \mid I_t = y\} = \frac{\lim_{t\to\infty} P\{R_t \leqq x, I_t = y\}}{\lim_{t\to\infty} P\{I_t = y\}}. \qquad (*)$$

By (7.20),

$$\lim_{t\to\infty} P\{R_t \leqq x, I_t = y\} = \frac{1}{\beta}\int_0^x dG(y)\, d\xi = \frac{x\, dG(y)}{\beta},$$

and setting $x = y$ in the above equation we derive

$$\lim_{t\to\infty} P\{I_t = y\} = \frac{y\, dG(y)}{\beta}.$$

Substitution of the last two expressions into $(*)$ proves (7.21) in the discontinuous case.

Next we prove (7.21) under the assumption that $G(\xi)$ is continuous and differentiable at $\xi = y$. Then

$$\lim_{t\to\infty} P\{R_t \leqq x \mid I_t = y\} = \lim_{t\to\infty} P\{R_t \leqq x \mid y \leqq I_t < y + dy\}$$

$$= \frac{\lim_{t\to\infty} P\{R_t \leqq x, y \leqq I_t < y + dy\}}{\lim_{t\to\infty} P\{y \leqq I_t < y + dy\}} \qquad (**)$$

By (7.20),

$$\lim_{t\to\infty} P\{R_t \leqq x, y \leqq I_t < y + dy\} = \left(\frac{1}{\beta}\int_0^x \frac{dG(y)}{dy}\, d\xi\right) dy = \frac{x}{\beta}\, \frac{dG(y)}{dy}\, dy,$$

and by setting $x = y$ in this equation we find

$$\lim_{t\to\infty} P\{y \leqq I_t < y + dy\} = \frac{y}{\beta}\, \frac{dG(y)}{dy}\, dy.$$

Substitution into $(**)$ once more results in (7.21).

(Chap. 5, Ex. 9 e)

$\boxed{e}$ The equivalence of (7.20) and (7.22) is established by the following sequence of pairwise equivalent equations leading from (7.20) to (7.22):

$$\lim_{t\to\infty} P\{R_t \leqq x, I_t \leqq y\} = \frac{1}{\beta}\int_0^x [G(y)-G(\xi)]\,d\xi \qquad (0 \leqq x \leqq y) \qquad (7.20)$$

$$\lim_{t\to\infty} d\,P\{R_t \leqq x, I_t \leqq y\} = \frac{1}{\beta}\,d\,G(y)\,dx \qquad (0 \leqq x \leqq y)$$

$$\lim_{t\to\infty} d\,P\{R_t \leqq x, A_t \leqq y\} = \frac{1}{\beta}\,d\,G(x+y)\,dx$$

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \int_x^\infty \int_{y+}^\infty \frac{1}{\beta}\,dG(\xi+\eta)\,d\xi$$

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \frac{1}{\beta}\int_x^\infty [1-G(\xi+y)]\,d\xi$$

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = \frac{1}{\beta}\int_{x+y}^\infty [1-G(\xi)]\,d\xi \qquad (7.22)$$

$\boxed{f}$ Observe that, by (7.22), the probability that either $R_t = 0$ or $A_t = 0$ is zero, since $\lim\limits_{t\to\infty} P\{R_t > 0, A_t > 0\} = \frac{1}{\beta}\int_0^\infty [1-G(\xi)]\,d\xi = 1$.

Suppose now the renewal process is a Poisson process, that is, $G(\xi) = 1 - e^{-\beta^{-1}\xi}$. Substituting $G(\xi)$ in (7.22) it is found that

$$\lim_{t\to\infty} P\{R_t > x, A_t > y\} = e^{-\beta^{-1}(x+y)} \qquad (*)$$

Now,

$$\lim_{t\to\infty} P\{R_t > x\} = \lim_{t\to\infty} P\{R_t > x, A_t \geq 0\}$$
$$= \lim_{t\to\infty} P\{R_t > x, A_t > 0\}.$$

By (*),

$$\lim_{t\to\infty} P\{R_t > x\} = e^{-\beta^{-1}x}.$$

Similarly,

$$\lim_{t\to\infty} P\{A_t > y\} = e^{-\beta^{-1}y}.$$

As $\lim\limits_{t\to\infty} P\{R_t > x, A_t > y\} = \lim\limits_{t\to\infty} P\{R_t > x\} \lim\limits_{t\to\infty} P\{A_t > y\}$, the conclusion is that R_t and A_t are, in the limit, independent exponential variables. □

Chapter 5, Exercise 10

'Customers arrive at a single server...'

Blocked customers cleared.
$G(t)$ = interarrival time distribution function
$H(t)$ = service-time distribution function
$F(x)$ = cycle-time distribution function

$\boxed{a}$

$$F(x) = \int_0^x P\{R_t \leq x-t\}\, dH(t),$$

where $t=0$ is the time service starts. By Eq. (7.14),

$$F(x) = \int_0^x \Big(\sum_{j=1}^{\infty} \int_t^x [1-G(x-y)]\, dG^{*j}(y)\Big)\, dH(t).$$

Interchanging the order of integration and summation we find

$$F(x) = \sum_{j=1}^{\infty} \int_0^x \int_t^x [1-G(x-y)]\, dG^{*j}(y)\, dH(t). \qquad (1)$$

Henceforth we assume $H(t) = 1 - e^{-\mu t}$ $(t \geq 0)$.

$\boxed{b}$ Substitution of $dH(t) = \mu e^{-\mu t} dt$ and change of the order of integration give

$$F(x) = \sum_{j=1}^{\infty} \int_{y=0}^{x} [1-G(x-y)] \Big[\int_{t=0}^{y} \mu e^{-\mu t} dt\, dG^{*j}(y).$$

Hence,

$$F(x) = \sum_{j=1}^{\infty} \int_0^x [1-G(x-y)][1-e^{-\mu y}]\, dG^{*j}(y).$$

By differentiation w.r.t. x we find

$$dF(x) = \sum_{j=1}^{\infty} [1-G(0)][1-e^{-\mu x}]\, dG^{*j}(x) - \sum_{j=1}^{\infty} \int_0^x dG(x-y)[1-e^{-\mu y}]\, dG^{*j}(y).$$

With $G(0)=0$, this expression can be written

$$dF(x) = \sum_{j=1}^{\infty} dG^{*j}(x) - \sum_{j=1}^{\infty} e^{-\mu x} dG^{*j}(x) - \sum_{j=1}^{\infty} dG^{*(j+1)}(x) + \sum_{j=1}^{\infty} d\tilde{G}_j(x),$$

where

$$d\tilde{G}_j(x) = \int_0^x dG(x-y) \cdot e^{-\mu y}\, dG^{*j}(y).$$

(Chap. 5, Ex. 10 b)

Hence,

$$\begin{aligned}\phi(s) &= \int_0^\infty e^{-sx} dF(x) \\ &= \sum_{j=1}^{\infty} \int_0^\infty e^{-sx} d\,G^{*j}(x) - \sum_{j=1}^{\infty} \int_0^\infty e^{-(s+\mu)x} dG^{*j}(x) \\ &\quad - \sum_{j=1}^{\infty} \int_0^\infty e^{-sx} d\,G^{*(j+1)}(x) + \sum_{j=1}^{\infty} \int_0^\infty e^{-sx} d\tilde{G}_j(x)\end{aligned}$$

Introducing $\gamma(s) = \int_0^\infty e^{-st} dG(t)$ we derive

$$\phi(s) = \sum_{j=1}^{\infty} [\gamma(s)]^j - \sum_{j=1}^{\infty} [\gamma(s+\mu)]^j - \sum_{j=1}^{\infty} [\gamma(s)]^{j+1} + \sum_{j=1}^{\infty} \gamma(s)[\gamma(s+\mu)]^j,$$

which reduces to

$$\phi(s) = \frac{\gamma(s) - \gamma(s+\mu)}{1 - \gamma(s+\mu)}. \tag{2}$$

c

$$\phi'(s) = \frac{\gamma'(s)}{1-\gamma(s+\mu)} - (1-\gamma(s)) \frac{\gamma'(s+\mu)}{(1-\gamma(s+\mu))^2}.$$

The mean cycle time α is given by $\alpha = -\phi'(0)$. Thus,

$$\alpha = \frac{-\gamma'(0)}{1-\gamma(\mu)}. \tag{3}$$

d Evidently, the equilibrium probability P that the server is busy equals the ratio of mean service-time to mean cycle-time. That is $P = \mu^{-1}/\alpha$. By (3),

$$P = [-\gamma'(0)\mu]^{-1}[1-\gamma(\mu)]. \tag{4}$$

e By Eq. (1.4) of Chapter 2, the blocking probability is $\Pi = E(N)/(1+E(N))$, where N is the number of arrivals during a random service initiated by an arrival at an idle server. N is the number of failures in a sequence of Bernoulli trials. A failure is the occurrence of another arrival before service completion and has probability $q = 1-p = \int_0^\infty e^{-\mu t} dG(t) = \gamma(\mu)$. Thus N has the geometric distribution and $E(N) = q/p = \gamma(\mu)/(1-\gamma(\mu))$. Hence,

$$\Pi = \gamma(\mu). \tag{5}$$

(Chap. 5, Ex. 10 f)

$\boxed{f}$ Equations (4) and (5) imply

$$P = [-\gamma'(0)\mu]^{-1}[1-\Pi]. \qquad (6)$$

It is easy to see that $P(=P_1)$ equals the carried load. Now, $-\gamma'(0)$ is the mean interarrival time. Hence, letting λ = mean arrival rate, $\lambda = [-\gamma'(0)]^{-1}$. Thus, $P = \frac{\lambda}{\mu}(1-\Pi)$. That is, carried load (P) equals offered load ($\frac{\lambda}{\mu}$) times acceptance probability $(1-\Pi)$.

$\boxed{g}$ Suppose $G(t) = 1 - e^{-\lambda t}$ Then $\gamma(s) = \lambda/(\lambda+s)$. By (5) and (6),

$$\Pi = \frac{\lambda}{\lambda+\mu},$$

$$P = \frac{\lambda}{\mu}(1-\Pi) = \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\lambda+\mu}\right) = \frac{\lambda}{\lambda+\mu}.$$

In this case, then, (i) $P = \Pi$, as anticipated. By (2),

$$\phi(s) = \frac{\gamma(s) - \gamma(s+\mu)}{1 - \gamma(s+\mu)} = \frac{\frac{\lambda}{\lambda+s} - \frac{\lambda}{\lambda+s+\mu}}{1 - \frac{\lambda}{\lambda+s+\mu}} = \frac{\lambda}{\lambda+s}\cdot\frac{\mu}{\mu+s}.$$

Since $\phi(s)$ is the product of two Laplace transforms of exponential distributions we can conclude that statement (ii) is true.

$\boxed{h}$ Suppose interarrival times are of constant length τ. Then

$$F(x) = 1 - e^{-\mu j\tau} \qquad (j\tau \leq x < (j+1)\tau;\ j = 0,1,2,\ldots), \qquad (7)$$

and

$$\phi(s) = \int_0^\infty e^{-sx} dF(x) = \sum_{j=1}^\infty e^{-sj\tau}[e^{-\mu(j-1)\tau} - e^{-\mu j\tau}]$$

$$= (e^{\mu\tau}-1)\sum_{j=1}^\infty [e^{-(s+\mu)\tau}]^j = \frac{e^{-s\tau} - e^{-(s+\mu)\tau}}{1 - e^{-(s+\mu)\tau}}.$$

We would get the same result from an application of Eq. (2). In the present case, $\gamma(s) = \int_0^\infty e^{-st}dG(t) = e^{-s\tau}$. Substitution into (2) yields the formula above.

□

Chapter 5, Exercise 11

'Customers arrive according to a renewal process...'

$\boxed{a}$ Suppose the arrival stream at a server is a renewal process, and that service times are exponential, and blocked customers are cleared. Let $\{T_j\}$ $(j=1,2,\ldots)$ denote the resultant sequence of overflow epochs.

$T_{j+1}-T_j$ is completely determined by (i) remaining service time at T_j, (ii) the sequence of future service times, (iii) the sequence of future interarrival intervals. All these variables are independent of the process up until T_j, and also independent of j. It follows that the sequence of interevent intervals $\{T_{j+1}-T_j\}$ $(j=1,2,\ldots)$ are independent, identically distributed random variables. That is, the overflow stream is a renewal process.

We conclude that, under the assumptions of this exercise, the overflow stream from the i'th ordered server $(i=1,2,\ldots)$ is a renewal process.

$\boxed{b}$ Let $G_i(t)$ be the distribution function of times between successive overflows from the i'th server. Choose an arbitrary overflow epoch of the i'th server. Let X be the time until next arrival (i.e. overflow from the (i-1)th server), and let Y be the time until next overflow from the i'th server. Then, for $x \leq t$,

$$P\{Y \leq t \mid X=x\} = e^{-\mu x} + (1-e^{-\mu x})G_i(t-x),$$

with $G_0(0)=0$, which implies $G_i(0)=0$. This is so, since for $X=x$ the event $Y \leq t$ will occur if (i) service is completed after next arrival (x time units later), having probability $e^{-\mu x}$, or if (ii) service is completed before next arrival and an overflow from the i'th server takes place during the time interval $(x,t]$, having probability $(1-e^{-\mu x})G_i(t-x)$. Observe that the time from start of service until next overflow from the i'th server has the same distribution as the inter-overflow times of the server.

Clearly,

$$P\{Y \leq t\} = \int_0^t P\{Y \leq t \mid X=x\}\, dP\{X \leq x\}.$$

(Chap. 5, Ex. 11 b)

Now, $P\{Y \leqq t\} = G_i(t)$ and $P\{X \leqq x\} = G_{i-1}(x)$. Hence,

$$G_i(t) = \int_0^t \left[e^{-\mu x} + (1 - e^{-\mu x})\, G_i(t-x)\right] dG_{i-1}(x) \qquad (i = 1,2,\dots) \qquad (1)$$

as asserted.

$\boxed{c}$ Differentiation of Eq. (1) leads to

$$dG_i(t) = e^{-\mu t} dG_{i-1}(t) + \int_0^t (1 - e^{-\mu x})\, dG_i(t-x)\, dG_{i-1}(x).$$

It follows that

$$\begin{aligned}
\gamma_i(s) &= \int_0^\infty e^{-st} dG_i(t) \\
&= \int_0^\infty e^{-(s+\mu)t} dG_{i-1}(t) \\
&\quad + \int_0^\infty e^{-st} \int_0^t dG_i(t-x)\, dG_{i-1}(x) \\
&\quad - \int_0^\infty e^{-st} \int_0^t dG_i(t-x)\, e^{-\mu x} dG_{i-1}(x) \\
&= \int_0^\infty e^{-(s+\mu)t} dG_{i-1}(t) \\
&\quad + \int_0^\infty e^{-st} dG_i(t) \cdot \int_0^\infty e^{-st} dG_{i-1}(t) \\
&\quad - \int_0^\infty e^{-st} dG_i(t) \cdot \int_0^\infty e^{-st} e^{-\mu t} dG_{i-1}(t)
\end{aligned}$$

Hence,

$$\gamma_i(s) = \gamma_{i-1}(s+\mu) + \gamma_i(s)\gamma_{i-1}(s) - \gamma_i(s)\gamma_{i-1}(s+\mu),$$

from which is obtained the recurrence equation

$$\gamma_i(s) = \frac{\gamma_{i-1}(s+\mu)}{1 - \gamma_{i-1}(s) + \gamma_{i-1}(s+\mu)}. \qquad (i = 1,2,\dots) \qquad (2)$$

$\square$

Chapter 5, Exercise 12

'The M/G/1 queue with server vacation times.'

Let $P(j)$ be the probability that j customers arrive during a single, arbitrary vacation, and define

$$f(z) = \sum_{j=0}^{\infty} P(j)\, z^j .$$

Let X denote the number of customers $(X \geq 1)$ who arrive during the vacation(s). Clearly, $P\{X=0\}=0$ and, for $j \geq 1$, $P\{X=j\} = P(j)/[1-P(0)]$. Letting $\hat{f}(z) = \Sigma_{j=0}^{\infty} P\{X=j\}\, z^j$ be the probability generating function for X, we derive

$$\hat{f}(z) = \frac{f(z) - P(0)}{1 - P(0)} . \qquad (*)$$

[a] Proceeding as in the analysis of the M/G/1 queue <u>without</u> vacation, we find the following system of equations:

$$\hat{\Pi}_j^* = r_{j+1} \hat{\Pi}_0^* + \sum_{i=1}^{j+1} p_{j-i+1} \hat{\Pi}_i^* \quad (j = 0, 1, \ldots) . \qquad (8.20)'$$

Here, $r_{j+1} = P\{X+Y = j+1\}$, where X is the number of customers arriving during the vacation period and Y is the number of customers arriving during the initial service after vacation(s).

Consider for a moment the p.g.f. of $X+Y$. Since X and Y are independent variables

$$\sum_{j=0}^{\infty} r_j z^j = \hat{f}(z) \cdot h(z) , \qquad (**)$$

with $h(z) = \Sigma_{j=0}^{\infty} p_j z^j$ being the p.g.f. of the number of arrivals during an arbitrary service time. Note, $r_0 = 0$, as $X \geq 1$.

Substitution of Eq. (8.20)′ into the generating function

$$\hat{g}(z) = \sum_{j=0}^{\infty} \hat{\Pi}_j^* z^j$$

results in

$$\hat{g}(z) = \hat{\Pi}_0^* \sum_{j=0}^{\infty} r_{j+1} z^j + \sum_{j=0}^{\infty} \sum_{i=1}^{j+1} p_{j-i+1} \hat{\Pi}_i^* z^j . \qquad (8.23)'$$

(Chap. 5, Ex. 12 a)

Now, by (**) and the fact that $r_0 = 0$,

$$\sum_{j=0}^{\infty} r_{j+1} z^j = z^{-1} \sum_{j=0}^{\infty} r_j z^j = z^{-1} \hat{f}(z) h(z),$$

and, furthermore,

$$\sum_{j=0}^{\infty} \sum_{i=1}^{j+1} P_{j-i+1} \hat{\Pi}_i^* z^j = z^{-1} (\hat{g}(z) - \hat{\Pi}_0^*) h(z). \qquad (8.28)'$$

Hence,

$$\hat{g}(z) = \hat{\Pi}_0^* z^{-1} \hat{f}(z) h(z) + z^{-1} (\hat{g}(z) - \hat{\Pi}_0^*) h(z). \qquad (8.29)'$$

By (*), $\hat{f}(z) = 1 + (f(z)-1)/(1-P(0))$. Substituting this into (8.29)′ and simplifying, we obtain

$$\hat{g}(z) = \hat{\Pi}_0^* z^{-1} \frac{f(z)-1}{1-P(0)} h(z) + z^{-1} \hat{g}(z) h(z).$$

Solving for $\hat{g}(z)$, we find

$$\hat{g}(z) = \frac{(f(z)-1)\, h(z)}{z - h(z)} \cdot \frac{\hat{\Pi}_0^*}{1 - P_0}. \qquad (8.30)'$$

A utilization of the condition $\hat{g}(1) = 1$ gives

$$\hat{\Pi}_0^* = (1-\rho) \frac{1-P(0)}{f'(1)} \qquad (8.32)'$$

with $\rho = \lambda\tau = h'(1)$. When this expression and $h(z) = \eta(\lambda - \lambda z)$ are substituted into (8.30)′ we get

$$\hat{g}(z) = \frac{[f(z)-1]\, \eta(\lambda-\lambda z)}{z - \eta(\lambda - \lambda z)} \; \frac{1-\rho}{f'(1)}. \qquad (1)$$

A comparison of (1) and (8.12) shows that

$$\hat{g}(z) = \frac{f(z)-1}{f'(1)(z-1)} \times g(z) \qquad (2)$$

where $g(z)$ is the probability generating function of the number of customers left behind by an arbitrary departing customer in the corresponding equilibrium M/G/1 system, in which the server never goes on vacation.

(Chap. 5, Ex. 12 b)

$\boxed{b}$ We shall prove that $\hat{g}(z) = g(z) \Leftrightarrow f(z) = P(0) + P(1)z$. This means that $\hat{g}(z) = g(z)$ implies that no more than one customer will ever arrive during a vacation. As the arrival process is Poisson, $\hat{g}(z) = g(z)$ therefore automatically rules out the possibility that vacation length is independent of the arrival process. Our explanation of the condition $\hat{g}(z) = g(z)$ is that a vacation, if not already over, will be interrupted the moment an arrival takes place.

By (2), $\hat{g}(z) = g(z) \Leftrightarrow f(z) - 1 = f'(1)(z-1)$. Hence, it will suffice to show that $f(z) - 1 = f'(1)(z-1) \Leftrightarrow f(z) = P(0) + P(1)z$.

Necessity ($\Rightarrow$). Assume $f(z) - 1 = f'(1)(z-1)$ for all z. Then $f(z) = (1 - f'(1)) + f'(1)z$. Since $f(z) = \sum_{j=0}^{\infty} P(j) z^j$, it follows that $P(0) = 1 - f'(1)$ and $P(1) = f'(1)$. Thus, $f(z) = P(0) + P(1)z$.

Sufficiency ($\Leftarrow$). Assume $f(z) = P(0) + P(1)z$. Then $P(0) = 1 - P(1)$ so that $f(z) - 1 = P(1)(z-1)$. Hence $f'(z) = P(1)$ and, in particular, $f'(1) = P(1)$. Thus, $f(z) - 1 = f'(1)(z-1)$ for all z.

$\boxed{c}$ By Eq. (3.3), $\hat{\pi}_j = \hat{\pi}_j^*$ for all $j \geq 0$. Consequently,

$$\sum_{j=0}^{\infty} \hat{\pi}_j z^j = \sum_{j=0}^{\infty} \hat{\pi}_j^* z^j = \hat{g}(z),$$

where $\hat{g}(z)$ is given by (1). By (8.32)$'$ and $\hat{\pi}_0 = \hat{\pi}_0^*$,

$$\hat{\pi}_0 = (1-\rho)\frac{1-P(0)}{f'(1)}.$$

($f'(1)/(1-P(0))$ is mean number of customers by end of vacation(s).)

$\boxed{d}$ Differentiation of Eq. (2) gives

$$\hat{g}'(z) = \frac{f(z)-1}{f'(1)(z-1)} g'(z) + \frac{g(z)}{f'(1)} \frac{(z-1)f'(z) - (f(z)-1)}{(z-1)^2},$$

whereby

$$\hat{g}'(1) = \lim_{z \to 1} \frac{f(z)-1}{f'(1)(z-1)} g'(1) + \frac{1}{f'(1)} \lim_{z \to 1} \frac{(z-1)f'(z) - (f(z)-1)}{(z-1)^2}.$$

(Chap. 5, Ex. 12 d)

A single application of l'Hospital's rule yields

$$\hat{g}'(1) = \lim_{z\to 1} \frac{f'(z)}{f'(1)\cdot 1}\, g'(1) + \frac{1}{f'(1)} \lim_{z \to 1} \frac{(z-1) f''(z) + f'(z) - f'(z)}{2(z-1)}.$$

Hence,

$$\hat{g}'(1) = g'(1) + \frac{f''(1)}{2f'(1)}. \qquad (3)$$

[e] Letting $\hat{\phi}(s)$ be the Laplace-Stieltjes transform of the sojourn time, we have

$$\hat{\phi}(s) = \hat{\omega}(s)\,\eta(s), \qquad (8.42)'$$

$$\hat{g}(z) = \hat{\phi}(\lambda - \lambda z) \qquad (8.43)'$$

Thus,

$$\hat{g}(z) = \hat{\omega}(\lambda-\lambda z)\,\eta(\lambda-\lambda z).$$

Insertion of this expression into (1) leads to

$$\hat{\omega}(\lambda-\lambda z) = \frac{f(z)-1}{z-\eta(\lambda-\lambda z)}\, \frac{1-\rho}{f'(1)}. \qquad (8.44)'$$

Setting $s = \lambda - \lambda z$, we derive the analogue of (8.38)

$$\hat{\omega}(s) = \frac{1-f(1-\frac{s}{\lambda})}{s-\lambda[1-\eta(s)]}\, \frac{(1-\rho)\lambda}{f'(1)}. \qquad (4)$$

[f] In the case that vacation lengths are independent of the arrival process, evidently the probability of waiting equals 1.

[g] The L.-S. transform of the distribution function of the waiting time $\hat{W}$ is $\hat{\omega}(s)$, given by (4). By comparison of (4) and (8.38) we find the relation

$$\hat{\omega}(s) = \frac{1-f(1-\frac{s}{\lambda})}{s}\, \frac{\lambda}{f'(1)}\, \omega(s).$$

Hence,

$$\hat{\omega}'(s) = \frac{1-f(1-\frac{s}{\lambda})}{s}\, \frac{\lambda}{f'(1)}\, \omega'(s) + \frac{\frac{s}{\lambda} f'(1-\frac{s}{\lambda}) - (1-f(1-\frac{s}{\lambda}))}{s^2}\, \frac{\lambda}{f'(1)}\, \omega(s),$$

$$\hat{\omega}'(0) = \lim_{s \to 0} \frac{1-f(1-\frac{s}{\lambda})}{s} \frac{\lambda}{f'(1)} \omega'(0) + \lim_{s \to 0} \frac{\frac{s}{\lambda} f'(1-\frac{s}{\lambda}) - (1-f(1-\frac{s}{\lambda}))}{s^2} \frac{\lambda}{f'(1)}.$$

A single application of l'Hospitals rule produces

$$\hat{\omega}'(0) = \lim_{s \to 0} \frac{\frac{1}{\lambda} f'(1-\frac{s}{\lambda})}{1} \frac{\lambda}{f'(1)} \omega'(0) + \lim_{s \to 0} \frac{\frac{1}{\lambda} f'(1-\frac{s}{\lambda}) - \frac{s}{\lambda^2} f''(1-\frac{s}{\lambda}) - \frac{1}{\lambda} f'(1-\frac{s}{\lambda})}{2s} \frac{\lambda}{f'(1)},$$

by which

$$\hat{\omega}'(0) = \omega'(0) - \frac{f''(1)}{2\lambda f'(1)}.$$

Now, $E(\hat{W}) = -\hat{\omega}'(0)$ and $E(W) = -\omega'(0)$. It follows that

$$E(\hat{W}) = E(W) + \frac{f''(1)}{2\lambda f'(1)}, \qquad (5)$$

where $E(W)$ is given by (8.39).

$\boxed{h}$ Suppose $f(z) = z^j$ ($j \geq 1$). That is, with probability 1 exactly j customers arrive during a vacation, and arrival no. j signals the end of the vacation. Obviously, the length of a vacation depends on future arrivals.

Eqs. (1), (2), and (3), hold for $f(z) = z^j$ for all $j \geq 1$, since the equations were derived without a requirement of independence between vacation length and arrival process.

Eq. (4) results from (1), (8.42)′ and (8.43)′. Of these, (1) and (8.42)′ hold whether or not vacation length depends on the arrival process. However, (8.43)′ is valid only if sojourn time is independent of the arrival process. For $f(z) = z^j$ with $j = 1$ this condition will be met for every customer, but if $j \geq 2$, then in the case of arrivals no. $1, 2, \ldots, j-1$, the time until the end of vacation will depend on the future arrival epochs.

We conclude that Eq. (4) is valid for $j = 1$, but not for $j \geq 2$, when $f(z) = z^j$. In the case $f(z) = z$, (4) reduces to the Pollaczek-Khintchine formula (8.38), as it should, when $f(1-\frac{s}{\lambda}) = 1 - \frac{s}{\lambda}$ and $f'(1) = 1$ are inserted.

Eq. (5) follows from (3) after application of $L = \lambda W$; (5) is therefore valid for all $j \geq 1$ when $f(z) = z^j$. □

Chapter 5, Exercise 13

'Derivation of the P.-K. formula by the method of collective marks'

Note that in this exercise the same notation is used for a time interval and its length. For instance, W_k will denote both the time interval during which the k'th customer waits for service and the length of that time interval, the waiting time. No confusion should arise as the meaning is clear from the context.

$\boxed{a}$ We consider an M/G/1 queue with order-of-arrival service. The k'th arriving customer is here the same as the k'th departing customer, so we may also speak of the k'th customer.

Let $\Pi_j^{*(k)}$ be the probability that the k'th customer will leave j customers behind, namely those customers who arrive during his sojourn time, and define the generating function

$$g_k(z) = \sum_{j=0}^{\infty} \Pi_j^{*(k)} z^j .$$

Now, imagine that each arriving customer is marked with probability $1-z$ and left unmarked with probability z. Clearly, by the theorem of total probability, $g_k(z)$ may be interpreted as the probability that no marked customers arrive during the sojourn time of the k'th customer.

$\boxed{b}$ T_k = sojourn time of the k'th customer (or corresponding time interval)

W_k = waiting time of the k'th customer (or corresponding time interval)

$C_k = \{$ the k'th customer is marked $\}$

$C_k' = \{$ the k'th customer is not marked $\}$

$M'(X) = \{$ no marked customers arrive during time interval X $\}$

It follows from the above definitions and our assumption of order-of-arrival service that

$$\{M'(T_k), C_{k+1}\} \Leftrightarrow \{W_{k+1} = 0, C_{k+1}\},$$
$$\{M'(T_k), C_{k+1}'\} \Leftrightarrow \{M'(W_{k+1}), C_{k+1}'\}.$$

(Chap. 5, Ex. 13 b)

Also, since a customer's probability of being marked is $1-z$ whatever his waiting time and markings of other customers,

$$P\{W_{k+1}=0,\ C_{k+1}\} = P\{W_{k+1}=0\}\,P\{C_{k+1}\},$$
$$P\{M'(W_{k+1}),\ C'_{k+1}\} = P\{M'(W_{k+1})\}\,P\{C'_{k+1}\}.$$

Since $P\{M'(T_k)\} = P\{M'(T_k),\ C_{k+1}\} + P\{M'(T_k),\ C'_{k+1}\}$, we have

$$P\{M'(T_k)\} = P\{W_{k+1}=0\}(1-z) + P\{M'(W_{k+1})\}\,z.$$

$\boxed{c}$ Denote by $\phi_k(s)$ and $\omega_k(s)$ the Laplace-Stieltjes transforms of the distribution functions of T_k and W_k, respectively. By the definition of $g_k(z)$, the interpretation in part (a), and Equation (6.10),

$$P\{M'(T_k)\} = g_k(z) = \phi_k(\lambda-\lambda z).$$

Similarly,

$$P\{M'(W_{k+1})\} = \ \dots\ = \omega_{k+1}(\lambda-\lambda z).$$

$\boxed{d}$ By parts (b) and (c),

$$\phi_k(\lambda-\lambda z) = P\{W_{k+1}=0\}(1-z) + \omega_{k+1}(\lambda-\lambda z)\,z.$$

The substitution $\lambda-\lambda z = s$ gives

$$\phi_k(s) = P\{W_{k+1}=0\}\frac{s}{\lambda} + \omega_{k+1}(s)\left(1-\frac{s}{\lambda}\right).$$

Introducing $\phi_k(s)=\omega_k(s)\eta(s)$, letting $k\to\infty$, assuming that $\lim_{k\to\infty}\omega_k(s)=\omega(s)$ and $\lim_{k\to\infty}P\{W_k=0\} = P\{W=0\}$, and solving for $\omega(s)$, we find

$$\omega(s) = \frac{s}{s-\lambda[1-\eta(s)]}\,P\{W=0\}.$$

$\boxed{e}$ Finally, utilizing $\omega(0)=1$ we derive $P\{W=0\} = 1+\lambda\eta'(0) = 1-\lambda\tau = 1-\rho$. Once more we obtain the Pollaczek-Khintchine formula

$$\omega(s) = \frac{s(1-\rho)}{s-\lambda[1-\eta(s)]} \qquad (8.38).$$

$\square$

Chapter 5, Exercise 14

'The M/G/1 queue from the viewpoint of arrivals.'

Definitions

N = number of customers in the system just prior to an arbitrary arrival epoch T_c

R = remaining service time at T_c

$$\Pi_j = P\{N = j\} \qquad (j = 0, 1, \ldots) \tag{1}$$

$$\Pi_j(x) = P\{R \leq x, N = j\} \qquad (x \geq 0;\ j = 1, 2, \ldots) \tag{2}$$

$$\psi_j(s) = \int_0^\infty e^{-sx} d\,\Pi_j(x) \qquad (j = 1, 2, \ldots) \tag{3}$$

$$u(s, z) = \sum_{j=1}^{\infty} \psi_j(s)\, z^j \tag{4}$$

Main results

$$u(s, z) = \frac{\Pi_0 \lambda z (1 - z)}{z - \eta(\lambda - \lambda z)} \; \frac{\eta(s) - \eta(\lambda - \lambda z)}{s - \lambda(1 - z)} \qquad \text{[see parts e-j]}, \tag{5}$$

$$\Pi_0 = 1 - \rho \qquad (\rho = \lambda\tau < 1) \qquad \text{[see part a]}, \tag{6}$$

where $\eta(s)$ is the Laplace-Stieltjes transform of the service-time distribution function $H(x)$, τ is the mean service time, and λ is the customer arrival rate. Inversion of (5) gives

$$\sum_{j=1}^{\infty} \Pi_j(x) z^j = \frac{(1 - \rho)\lambda z (1 - z)}{\eta(\lambda - \lambda z) - z} \int_0^\infty e^{-\lambda(1-z)\xi} \left[H(\xi + x) - H(\xi) \right] d\xi \qquad \text{[see part k]}. \tag{7}$$

$\boxed{a}$

$$u(0,1) = \Pi_0 \lambda \lim_{z \to 1} \frac{1 - z}{z - \eta(\lambda - \lambda z)} \lim_{s \to 0} \frac{\eta(s) - 1}{s}$$

$$= \Pi_0 \lambda \frac{-\eta'(0)}{1 + \lambda \eta'(0)} = \Pi_0 \frac{\lambda\tau}{1 - \lambda\tau} = \Pi_0 \frac{\rho}{1 - \rho}.$$

$$1 = \Pi_0 + \sum_{j=1}^{\infty} \Pi_j = \Pi_0 + \sum_{j=1}^{\infty} \psi_j(0)$$

$$= \Pi_0 + u(0, 1) = \Pi_0 \left(1 + \frac{\rho}{1 - \rho}\right).$$

Thus, Eq. (6), $\Pi_0 = 1 - \rho$, follows from Eq. (5) and $\sum \Pi_j = 1$.

(Chap. 5, Ex. 14 b)

$\boxed{b}$ By Eq. (7),

$$\sum_{j=1}^{\infty} \Pi_j(x) = \lim_{z\to 1} \sum_{j=1}^{\infty} \Pi_j(x) z^j$$
$$= (1-\rho)\lambda \lim_{z\to 1} \frac{1-z}{\eta(\lambda-\lambda z)-z} \int_0^{\infty} [H(\xi+x)-H(\xi)]\,d\xi$$
$$= (1-\rho)\lambda \frac{1}{1+\lambda\eta'(0)} \left(\int_0^{\infty}[1-H(\xi)]\,d\xi - \int_x^{\infty}[1-H(\xi)]\,d\xi \right).$$

Hence,
$$\sum_{j=1}^{\infty} \Pi_j(x) = \lambda \int_0^x [1-H(\xi)]\,d\xi,$$
so that
$$P\{R \leq x \mid N \geq 1\} = \frac{P\{R\leq x, N\geq 1\}}{P\{N\geq 1\}} = \frac{\sum_{j=1}^{\infty}\Pi_j(x)}{\sum_{j=1}^{\infty}\Pi_j} = \frac{\lambda\int_0^x[1-H(\xi)]\,d\xi}{\lambda\int_0^{\infty}[1-H(\xi)]\,d\xi}.$$

Since the mean service time may be expressed as $\tau = \int_0^{\infty}[1-H(\xi)]\,d\xi$,
$$P\{R \leq x \mid N \geq 1\} = \frac{1}{\tau}\int_0^x [1-H(\xi)]\,d\xi. \qquad (8)$$

This might have been anticipated by considering (7.9); disregarding idle periods, start-of-service epochs form a renewal process.

$\boxed{c}$ By Eq. (7),

$$\sum_{j=0}^{\infty} \Pi_j z^j = \Pi_0 + \lim_{x\to\infty} \sum_{j=1}^{\infty} \Pi_j(x) z^j$$
$$= \Pi_0 + \frac{(1-\rho)\lambda z(1-z)}{\eta(\lambda-\lambda z)-z} \int_0^{\infty} e^{-\lambda(1-z)\xi}[1-H(\xi)]\,d\xi$$
$$= \Pi_0 + \frac{(1-\rho)\lambda z(1-z)}{\eta(\lambda-\lambda z)-z} \left(\frac{1}{\lambda(1-z)} - \int_0^{\infty} e^{-\lambda(1-z)\xi} H(\xi)\,d\xi \right)$$
$$= \Pi_0 + \frac{(1-\rho) z}{\eta(\lambda-\lambda z)-z} \left(1 + \int_0^{\infty} H(\xi)\,d e^{-\lambda(1-z)\xi} \right)$$
$$= \Pi_0 + \frac{(1-\rho) z}{\eta(\lambda-\lambda z)-z} \left(1 - \int_{0-}^{\infty} e^{-\lambda(1-z)\xi}\,dH(\xi) \right)$$
$$= \Pi_0 + \frac{(1-\rho) z}{\eta(\lambda-\lambda z)-z} \left(1 - \eta(\lambda-\lambda z) \right).$$

Finally, substitution of $\Pi_0 = 1-\rho$, by (6), and simplification give as the result,

(Chap. 5, Ex. 14 c)

$$\sum_{j=0}^{\infty} \Pi_j z^j = \frac{(z-1)\eta(\lambda-\lambda z)}{z-\eta(\lambda-\lambda z)}(1-\rho). \tag{9}$$

Equation (9) is seen to be identical to (8.12) which gives the probability generating function of the departure state. This is the direct consequence of the equality $\Pi_j^* = \Pi_j$.

$\boxed{d}$ Let $W_j(x)$ be the conditional waiting time distribution function, given $N=j$, and let $W(x)$ be the unconditional waiting time distribution function, assuming service in order of arrival. Then

$$W(x) = \sum_{j=0}^{\infty} \Pi_j W_j(x) = \Pi_0 + \sum_{j=1}^{\infty} \Pi_j W_j(x) = \Pi_0 + \sum_{j=1}^{\infty} \Pi_j \int_0^x P\{R \le x-\xi \mid N=j\} dH^{*(j-1)}(\xi),$$

where we have used that $W_j(x)$ is the convolution of the distribution of the remaining service time, given $N=j$, and the distribution of a sum of $j-1$ independent service times. By definition of $\Pi_j(x)$, then,

$$W(x) = \Pi_0 + \sum_{j=1}^{\infty} \int_0^x \Pi_j(x-\xi)\, dH^{*(j-1)}(\xi). \tag{10}$$

Hence,

$$\begin{aligned}
\omega(s) &= \int_0^{\infty} e^{-sx} dW(x) = \Pi_0 + \sum_{j=1}^{\infty} \int_0^{\infty} e^{-sx} d\int_0^x \Pi_j(x-\xi) dH^{*(j-1)}(\xi) \\
&= \Pi_0 + \sum_{j=1}^{\infty} \left(\int_0^{\infty} e^{-sx} d\Pi_j(x)\right)\left(\int_0^{\infty} e^{-sx} dH^{*(j-1)}(x)\right) \\
&= \Pi_0 + \sum_{j=1}^{\infty} \psi_j(s)[\eta(s)]^{j-1} \\
&= \Pi_0 + \frac{1}{\eta(s)} u(s,\eta(s)) \\
&= \Pi_0 + \frac{1}{\eta(s)} \frac{\Pi_0 \lambda \eta(s)(1-\eta(s))}{s-\lambda(1-\eta(s))} \\
&= \Pi_0 \frac{s}{s-\lambda(1-\eta(s))}.
\end{aligned}$$

Thus, the Laplace-Stieltjes transform of the waiting time is

$$\omega(s) = \frac{s(1-\rho)}{s-\lambda[1-\eta(s)]}, \tag{11}$$

which, of course, is the same as Eq. (8.38).

(Chap. 5, Ex. 14e)

$\boxed{e}$ Let T_1 and T_2 be an arbitrary pair of consecutive arrival epochs, and let the associated arrival states be (N_1, R_1) and (N_2, R_2), respectively. Eqs. (12), (13) and (14) are derived on the assumption of identical state probability distributions at T_1 and T_2. The distribution at T_2, defined by Π_0 $(= P\{N_2 = 0\})$, $\Pi_1(x)$ $(= P\{N_2 = 1, R_2 \le x\})$ and $\Pi_j(x)$ $(= P\{N_2 = j, R_2 \le x\})$ for $j = 2, 3, \ldots$ is found by conditioning on (N_1, R_1). The equations reflect the fact that $N_2 = j$ may occur if and only if $N_1 \geq j-1$.

$$\Pi_0 = \Pi_0 \int_0^\infty e^{-\lambda t} dH(t) + \sum_{k=1}^{\infty} \int_0^\infty \int_0^\infty e^{-\lambda(y+z)} d\Pi_k(y)\, dH^{*k}(z) . \qquad (12)$$

Eq. (12) is obtained by rewriting $P\{N_2 = 0\} = P\{N_1 = 0, N_2 = 0\} + \sum_{k=1}^{\infty} P\{N_1 = k, N_2 = 0\}$ as $\Pi_0 = \Pi_0 P\{N_2 = 0 | N_1 = 0\} + \sum_{k=1}^{\infty} \int_0^\infty P\{N_2 = 0 | N_1 = k, R_1 = y\}\, d\Pi_k(y)$, and observing that $P\{N_2 = 0 | N_1 = 0\} = \int_0^\infty e^{-\lambda t} dH(t)$ and $P\{N_2 = 0 | N_1 = k, R_1 = y\} = \int_0^\infty e^{-\lambda(y+z)} dH^{*k}(z)$.

$$\Pi_1(x) = \Pi_0 \int_0^\infty [H(t+x) - H(t)]\lambda e^{-\lambda t} dt \qquad (13)$$
$$+ \sum_{k=1}^{\infty} \int_0^\infty \int_0^\infty e^{-\lambda(y+z)} d\Pi_k(y)\, dH^{*(k-1)}(z) \int_0^\infty [H(t+x) - H(t)]\lambda e^{-\lambda t} dt.$$

Eq. (13) follows from $P\{N_2 = 1, R_2 \le x\} = P\{N_1 = 0, N_2 = 1, R_2 \le x\} + \sum_{k=1}^{\infty} P\{N_1 = k, N_2 = 1, R_2 \le x\}$, rewritten $\Pi_1(x) = \Pi_0 P\{N_2 = 1, R_2 \le x | N_1 = 0\} + \sum_{k=1}^{\infty} \int_0^\infty P\{N_2 = 1, R_2 \le x | N_1 = k, R_1 = y\}\, d\Pi_k(y)$, as well as the observations that $P\{N_2 = 1, R_2 \le x | N_1 = 0\} = \int_0^\infty [H(t+x) - H(t)]\lambda e^{-\lambda t} dt$ and $P\{N_2 = 1, R_2 \le x | N_1 = k, R_1 = y\} = \int_0^\infty e^{-\lambda(y+z)} dH^{*(k-1)}(z) \int_0^\infty [H(t+x) - H(t)]\lambda e^{-\lambda t} dt$.

$$\Pi_j(x) = \int_0^\infty [\Pi_{j-1}(t+x) - \Pi_{j-1}(t)]\lambda e^{-\lambda t} dt \qquad (j = 2, 3, \ldots) \qquad (14)$$
$$+ \sum_{k=j}^{\infty} \int_0^\infty \int_0^\infty e^{-\lambda(y+z)} d\Pi_k(y)\, dH^{*(k-j)}(z) \int_0^\infty [H(t+x) - H(t)]\lambda e^{-\lambda t} dt.$$

Eq. (14) may be derived by the same kind of arguments that were used for Eqs. (12) and (13).

$\boxed{f}$ The double integral of Eq. (12) equals $\int_0^\infty e^{-\lambda y} d\Pi_k(y) \int_0^\infty e^{-\lambda z} dH^{*k}(z)$. Hence, using the Laplace-Stieltjes transform definitions, Eq. (12) becomes

$$\Pi_0 = \Pi_0\, \eta(\lambda) + \sum_{k=1}^{\infty} \psi_k(\lambda)\, \eta^k(\lambda). \qquad (15)$$

(Chap. 5, Ex. 14 f)

Eq. (13) may be written

$$\Pi_1(x) = K_1(x)\Big[\Pi_0 + \sum_{k=1}^{\infty} \psi_k(\lambda)\eta^{k-1}(\lambda)\Big],$$

where

$$K_1(x) = \int_0^{\infty} [H(t+x) - H(t)]\lambda e^{-\lambda t} dt.$$

Hence,

$$\psi_1(s) = \int_0^{\infty} e^{-sx} d\,\Pi_1(x) = \int_0^{\infty} e^{-sx} dK_1(x)\Big[\Pi_0 + \sum_{k=1}^{\infty} \psi_k(\lambda)\eta^{k-1}(\lambda)\Big].$$

Now,

$$\int_0^{\infty} e^{-sx} dK_1(x) = \int_{x=0}^{\infty} e^{-sx}\Big(\int_{t=0}^{\infty} \lambda e^{-\lambda t} dH(t+x)\Big)dx$$

$$= \lambda \int_{x=0}^{\infty} e^{-(s-\lambda)x}\Big(\int_{t=0}^{\infty} e^{-\lambda(t+x)} dH(t+x)\Big)dx = \lambda \int_{x=0}^{\infty} e^{-(s-\lambda)x}\Big(\int_{t=x}^{\infty} e^{-\lambda t} dH(t)\Big)dx$$

$$= \lambda \int_{t=0}^{\infty} e^{-\lambda t}\Big(\int_{x=0}^{t} e^{-(s-\lambda)x}dx\Big)dH(t) = \frac{\lambda}{s-\lambda}\int_{t=0}^{\infty} e^{-\lambda t}[1-e^{-(s-\lambda)t}]dH(t),$$

so that

$$\int_0^{\infty} e^{-sx} dK_1(x) = \frac{\lambda}{s-\lambda}[\eta(\lambda) - \eta(s)].$$

Hence,

$$\psi_1(s) = \frac{\lambda}{s-\lambda}[\eta(\lambda) - \eta(s)]\Big[\Pi_0 + \sum_{k=1}^{\infty} \psi_k(\lambda)\eta^{k-1}(\lambda)\Big]. \qquad (16)$$

Eq. (14) may be written

$$\Pi_j(x) = K_j(x) + K_1(x)\sum_{k=j}^{\infty} \psi_k(\lambda)\eta^{k-j}(\lambda) \qquad (j = 2,3,\dots),$$

where $K_1(x)$ has been defined above and

$$K_j(x) = \int_0^{\infty} [\Pi_{j-1}(t+x) - \Pi_{j-1}(t)]\lambda e^{-\lambda t} dt \qquad (j = 2,3,\dots).$$

Hence,

$$\psi_j(s) = \int_0^{\infty} e^{-sx} d\,\Pi_j(x) = \int_0^{\infty} e^{-sx} dK_j(x) + \int_0^{\infty} e^{-sx} dK_1(x)\sum_{k=j}^{\infty} \psi_k(\lambda)\eta^{k-j}(\lambda) \quad (j=2,3,\dots).$$

Proceeding precisely as when $\int_0^{\infty} e^{-sx} dK_1(x)$ was calculated, we derive

$$\int_0^{\infty} e^{-sx} dK_j(x) = \frac{\lambda}{s-\lambda}[\psi_{j-1}(\lambda) - \psi_{j-1}(s)] \qquad (j = 2,3,\dots).$$

Hence, for $j=2,3,\dots$,

$$\psi_j(s) = \frac{\lambda}{s-\lambda}[\psi_{j-1}(\lambda) - \psi_{j-1}(s)] + \frac{\lambda}{s-\lambda}[\eta(\lambda) - \eta(s)]\sum_{k=j}^{\infty} \psi_k(\lambda)\eta^{k-j}(\lambda) \qquad (17)$$

(Chap. 5, Ex. 14 g)

$\boxed{g}$ Substitution of (16) and (17) into (4) leads to

$$u(s,z) = \frac{\lambda}{s-\lambda}\sum_{j=2}^{\infty}[\psi_{j-1}(\lambda) - \psi_{j-1}(s)]\,z^j + \frac{\lambda}{s-\lambda}[\eta(\lambda)-\eta(s)]\Big[\Pi_0 z + \sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\psi_k(\lambda)\eta^{k-j}(\lambda)\,z^j\Big].$$

Now,

$$\sum_{j=2}^{\infty}[\psi_{j-1}(\lambda) - \psi_{j-1}(s)]\,z^j = z\,[u(\lambda,z) - u(s,z)],$$

and

$$\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\psi_k(\lambda)\,\eta^{k-j}(\lambda)\,z^j = \sum_{k=1}^{\infty}\psi_k(\lambda)\,\eta^k(\lambda)\sum_{j=1}^{k}\Big(\frac{z}{\eta(\lambda)}\Big)^j$$

$$= \frac{z}{\eta(\lambda)-z}\sum_{k=1}^{\infty}\psi_k(\lambda)\,\eta^k(\lambda)\Big[1-\Big(\frac{z}{\eta(\lambda)}\Big)^k\Big]$$

$$= \frac{z}{\eta(\lambda)-z}\,[u(\lambda,\eta(\lambda)) - u(\lambda,z)].$$

It follows easily that

$$u(s,z)[s-\lambda(1-z)] = \lambda z\,u(\lambda,z) + \lambda z[\eta(\lambda)-\eta(s)]\Big[\Pi_0 + \frac{u(\lambda,\eta(\lambda)) - u(\lambda,z)}{\eta(\lambda)-z}\Big]. \quad (18)$$

$\boxed{h}$ For $s = \lambda - \lambda z$, Eq. (18) specializes to

$$0 = \lambda z\,u(\lambda,z) + \lambda z[\eta(\lambda)-\eta(\lambda-\lambda z)]\Big[\Pi_0 + \frac{u(\lambda,\eta(\lambda)) - u(\lambda,z)}{\eta(\lambda)-z}\Big],$$

whereby

$$\Pi_0 = \frac{u(\lambda,z)}{\eta(\lambda-\lambda z)-\eta(\lambda)} + \frac{u(\lambda,\eta(\lambda)) - u(\lambda,z)}{z-\eta(\lambda)}. \quad (19)$$

Substitution of this expression into (18) and solution w.r.t. $u(s,z)$ give us

$$u(s,z) = \frac{\lambda z[\eta(s) - \eta(\lambda-\lambda z)]\,u(\lambda,z)}{[s-(\lambda-\lambda z)][\eta(\lambda)-\eta(\lambda-\lambda z)]}. \quad (20)$$

$\boxed{i}$ By (15),

$$u(\lambda,\eta(\lambda)) = \Pi_0\,[1-\eta(\lambda)]. \quad (21)$$

Substitution of this expression into (19) and solution w.r.t. $u(\lambda,z)$ give us

$$u(\lambda,z) = \frac{\Pi_0(1-z)[\eta(\lambda)-\eta(\lambda-\lambda z)]}{z-\eta(\lambda-\lambda z)}. \quad (22)$$

(Chap. 5, Ex. 14 j)

$\boxed{j}$ Finally, substitution of (22) into (20) yields

$$u(s,z) = \frac{\Pi_0 \lambda z(1-z)}{z-\eta(\lambda-\lambda z)} \; \frac{\eta(s)-\eta(\lambda-\lambda z)}{s-\lambda(1-z)} . \tag{5}$$

$\boxed{k}$ We shall show that inversion of (5) gives

$$\sum_{j=1}^{\infty} \Pi_j(x) z^j = A(z) \int_0^{\infty} e^{-\lambda(1-z)\xi} [H(\xi+x) - H(\xi)] \, d\xi , \tag{7}$$

where

$$A(z) = \frac{(1-\rho)\lambda z(1-z)}{\eta(\lambda-\lambda z)-z} .$$

To begin, we show that the Laplace-Stieltjes transform of the LHS of (7) equals the LHS of (5):

$$\int_{x=0}^{\infty} e^{-sx} d\Big(\sum_{j=1}^{\infty} \Pi_j(x) z^j\Big) = \sum_{j=1}^{\infty} z^j \int_{x=0}^{\infty} e^{-sx} d\Pi_j(x) = \sum_{j=1}^{\infty} \psi_j(s) z^j = u(s,z).$$

Next we show that the Laplace-Stieltjes transform of the RHS of (7) equals the RHS of (5):

$$\begin{aligned}
\int_{x=0}^{\infty} e^{-sx} d\Big(A(z)\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi}[H(\xi+x)-H(\xi)]d\xi\Big) &= A(z)\int_{x=0}^{\infty} e^{-sx}\Big(\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi} dH(\xi+x)\Big)dx \\
&= A(z)\int_{x=0}^{\infty} e^{-[s-\lambda(1-z)]x}\Big(\int_{\xi=0}^{\infty} e^{-\lambda(1-z)(\xi+x)} dH(\xi+x)\Big)dx \\
&= A(z)\int_{x=0}^{\infty} e^{-[s-\lambda(1-z)]x}\Big(\int_{\xi=x}^{\infty} e^{-\lambda(1-z)\xi} dH(\xi)\Big)dx \\
&= A(z)\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi}\Big(\int_{x=0}^{\xi} e^{-[s-\lambda(1-z)]x}dx\Big) dH(\xi) \\
&= A(z)\,\frac{1}{s-\lambda(1-z)}\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi}\big[1-e^{-[s-\lambda(1-z)]\xi}\big]dH(\xi) \\
&= A(z)\,\frac{1}{s-\lambda(1-z)}\int_{\xi=0}^{\infty}\big[e^{-\lambda(1-z)\xi}-e^{-s\xi}\big]dH(\xi) \\
&= A(z)\,\frac{1}{s-\lambda(1-z)}\big[\eta(\lambda-\lambda z)-\eta(s)\big]
\end{aligned}$$

Thus, considering the definition of $A(z)$ and the fact, by (6), that $1-\rho = \Pi_0$,

$$\int_{x=0}^{\infty} e^{-sx} d\Big(A(z)\int_{\xi=0}^{\infty} e^{-\lambda(1-z)\xi}[H(\xi+x)-H(\xi)]d\xi\Big) = \frac{\Pi_0 \lambda z(1-z)}{z-\eta(\lambda-\lambda z)} \; \frac{\eta(s)-\eta(\lambda-\lambda z)}{s-\lambda(1-z)} .$$

□

Chapter 5, Exercise 15

'The integrodifferential equation of Takács.'

Let V_t be the virtual waiting time in the M/G/1 queue with order-of-arrival service, and define the distribution function $V(t,x) = P\{V_t \leq x\}$.

$\boxed{a}$ Let $h>0$, and let K be the number of arrivals in the time interval $[t,t+h)$. We shall show that

$$P\{V_{t+h} \leq x,\ K=0\} = (1-\lambda h)\,V(t,x+h) + o(h), \qquad (i)$$

$$P\{V_{t+h} \leq x,\ K=1\} = \lambda h \int_0^{x+h} H(x+h-y)\,d_yV(t,y) + o(h), \qquad (ii)$$

whereby, as $P\{V_{t+h} \leq x,\ K\geq 2\} = o(h)$ and $V(t+h,x) = \sum_{j=0}^{\infty} P\{V_{t+h} \leq x,\ K=j\}$,

$$V(t+h,x) = (1-\lambda h)\,V(t,x+h) + \lambda h \int_0^{x+h} H(x+h-y)\,d_yV(t,y) + o(h). \qquad (*)$$

Eq. (i). Obviously, $\{V_{t+h} \leq x \mid K=0\} \Leftrightarrow \{V_t \leq x+h \mid K=0\}$. Consequently, $P\{V_{t+h} \leq x \mid K=0\} = P\{V_t \leq x+h \mid K=0\}$. Also, since V_t and K are independent, $P\{V_t \leq x+h \mid K=0\} = P\{V_t \leq x+h\}$. Thus, $P\{V_{t+h} \leq x \mid K=0\} = P\{V_t \leq x+h\}$, and,

$$P\{V_{t+h} \leq x,\ K=0\} = P\{K=0\}P\{V_{t+h} \leq x \mid K=0\} = (1-\lambda h + o(h))\,P\{V_t \leq x+h\},$$

whereby (i) follows.

Eq. (ii). For $K=1$, denote by t^* the time of arrival and by Z the service time of the customer arriving in $[t,t+h)$. Observe that $\{V_{t+h} \leq x \mid K=1\} \Leftrightarrow \{V_t + Z \leq x+h,\ t+h-t^* \geq Z-x \mid K=1\}$. Hence, for all $x \geq 0$,

$$P\{V_{t+h} \leq x \mid K=1\} = P\{V_t + Z \leq x+h \mid K=1\} - P\{V_t+Z \leq x+h,\ t+h-t^* < Z-x \mid K=1\}.$$

Since V_t, Z and K are independent variables,

$$P\{V_t + Z \leq x+h \mid K=1\} = P\{V_t + Z \leq x+h\} = \int_0^{x+h} H(x+h-y)\,d_yV(t,y).$$

(Chap. 5, Ex. 15 a)

For the sake of brevity, define

$$F(x,h) = P\{V_t + Z \leq x+h,\ t+h-t^* < Z-x \mid K=1\}.$$

Clearly, t^* is uniformly distributed on $[t, t+h)$, independently of V_t and Z. Using this fact we derive

$$\begin{aligned} F(x,h) &= \int_{y=0}^{h} \int_{z=x+}^{x+h-y} \frac{z-x}{h}\, dH(z)\, d_yV(t,y) \\ &\leq \int_{y=0}^{h} \int_{z=x+}^{x+h-y} dH(z)\, d_yV(t,y) \\ &\leq \int_0^h d_yV(t,y) \int_{x+}^{x+h} dH(z) \\ &= V(t,h)\,[H(x+h) - H(x)]. \end{aligned}$$

Hence,

$$\lim_{h\to 0} F(x,h) = 0 \qquad (x \geq 0).$$

Evidently,

$$P\{V_{t+h} \leq x, K=1\} = P\{K=1\}\, P\{V_{t+h} \leq x \mid K=1\} = (\lambda h + o(h))\, P\{V_{t+h} \leq x \mid K=1\}.$$

Hence, by previous results,

$$\begin{aligned} P\{V_{t+h} \leq x, K=1\} &= (\lambda h + o(h))\Big[\int_0^{x+h} H(x+h-y)\, d_yV(t,y) - F(x,h)\Big] \\ &= \lambda h \int_0^{x+h} H(x+h-y)\, d_yV(t,y) + o(h). \end{aligned}$$

This concludes the proof of (ii). The proof of Eq. (*) is complete.

$\boxed{b}$ Subtracting $V(t,x)$ on both sides of (*), dividing through by h, and letting $h \to 0$, we obtain

$$\frac{\partial V(t,x)}{\partial t} = \frac{\partial V(t,x)}{\partial x} - \lambda V(t,x) + \lambda \int_0^x H(x-y)\, d_yV(t,y), \tag{1}$$

which is the integrodifferential equation of Takács.

(Chap. 5, Ex. 15 c)

$\boxed{c}$ Assuming $\lim_{t\to\infty} \frac{\partial V(t,x)}{\partial t} = 0$ and $\lim_{t\to\infty} V(t,x) = V(x)$, Eq. (1) becomes

$$\frac{dV(x)}{dx} = \lambda V(x) - \lambda \int_0^x H(x-y)\,dV(y) \qquad (x \geqq 0). \tag{2}$$

Now, define the Laplace-Stieltjes transform

$$\theta(s) = \int_0^\infty e^{-sx} dV(x).$$

All three terms in Eq. (2) are functions of x. The L.-S. transforms of LHS and RHS of the equation are, respectively,

$$\begin{aligned}
\int_0^\infty e^{-sx} d\,\frac{dV(x)}{dx} &= V'(0) + \int_{0+}^\infty e^{-sx} d\,\frac{dV(x)}{dx} \\
&= V'(0) + e^{-sx}\frac{dV(x)}{dx}\Big|_0^\infty + s\int_{0+}^\infty e^{-sx} dV(x) \\
&= V'(0) - V'(0) + s\Big[\int_0^\infty e^{-sx} dV(x) - V(0)\Big] \\
&= s\,[\theta(s) - V(0)],
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty e^{-sx} d\Big(\lambda V(x) - \lambda\int_0^x H(x-y)\,dV(y)\Big) &= \lambda\int_0^\infty e^{-sx} dV(x) - \lambda\int_0^\infty e^{-sx} d\Big(\int_0^x H(x-y)\,dV(y)\Big) \\
&= \lambda\,\theta(s) - \lambda\,\eta(s)\,\theta(s),
\end{aligned}$$

since $\int_0^x H(x-y)\,dV(y)$ is the convolution of distribution functions with L.-S. transforms $\eta(s)$ and $\theta(s)$.

Equating LHS and RHS transforms we obtain

$$\theta(s) - V(0) = \frac{\lambda}{s}\theta(s) - \frac{\lambda}{s}\eta(s)\theta(s). \tag{3}$$

$\boxed{d}$ Solving (3) for $\theta(s)$ gives

$$\theta(s) = \frac{sV(0)}{s - \lambda[1-\eta(s)]}.$$

Inserting $V(0) = 1-\rho$ as usual for the M/G/1 queue, we get (8.38), as we should.

$\square$

Chapter 5, Exercise 16

'Verify that (8.72) is the solution of (8.71).'

$$f_n^{(j)} = \begin{cases} e^{-\lambda\tau j} & (n=j), \\ \sum_{k=1}^{n-j} \frac{(\lambda\tau j)^k}{k!} e^{-\lambda\tau j} f_{n-j}^{(k)} & (n \geqq j+1). \end{cases} \quad (8.71)$$

$$f_n^{(j)} = \frac{j}{n}\left[\frac{(\lambda\tau n)^{n-j}}{(n-j)!} e^{-\lambda\tau n}\right] \qquad (n \geqq j). \quad (8.72)$$

The proof is by induction. First, we observe that for all feasible n and j such that $n-j=0$, Eq. (8.72) reduces to $e^{-\lambda\tau n}$, which agrees with (8.71). Next, we assume that (8.72) already has been proved for all n and j such that $n-j \leqq k_0$. We shall show that then (8.72) will hold for all n and j such that $n-j=k_0+1$.

Thus, assume values of n and j such that $n-j=k_0+1$. By the induction hypothesis, (8.72) applies to all the factors $f_{n-j}^{(k)}$, $k=1,\ldots,n-j$, of (8.71), since $n-j-k \leqq k_0$. Substitution of (8.72) into (8.71) and straightforward reduction produce

$$\begin{aligned} f_n^{(j)} &= \sum_{k=1}^{n-j} \frac{(\lambda\tau j)^k}{k!} e^{-\lambda\tau j} \frac{k}{n-j} \frac{(\lambda\tau(n-j))^{n-j-k}}{(n-j-k)!} e^{-\lambda\tau(n-j)} \\ &= \frac{(\lambda\tau(n-j))^{n-j}}{(n-j)!} e^{-\lambda\tau n} \sum_{k=1}^{n-j} \frac{(n-j-1)!}{(k-1)!(n-j-k)!} \left(\frac{j}{n-j}\right)^k \\ &= \frac{(\lambda\tau(n-j))^{n-j}}{(n-j)!} e^{-\lambda\tau n} \frac{j}{n-j} \sum_{\nu=0}^{n-j-1} \binom{n-j-1}{\nu} \left(\frac{j}{n-j}\right)^{\nu} 1^{n-j-1-\nu} \qquad [\nu=k-1] \\ &= \frac{(\lambda\tau(n-j))^{n-j}}{(n-j)!} e^{-\lambda\tau n} \frac{j}{n-j} \left(1+\frac{j}{n-j}\right)^{n-j-1} \\ &= \frac{j}{n} \frac{(\lambda\tau n)^{n-j}}{(n-j)!} e^{-\lambda\tau n}. \end{aligned}$$

By induction, we conclude that (8.72) holds for all n and j where $j \geqq 1$, $n \geqq j$.

□

Chapter 5, Exercise 17

'a. Let N_k be the number of customers served during a k-busy period.'

$\boxed{a}$ Assume an M/G/1 queue. Starting at t_0 with $i+j$ customers in the system, we imagine that first we serve the i customers plus all later arrivals until the moment t_1, when a departure leaves the original j customers behind. Clearly, $[t_0, t_1)$ is an i-busy period. Let N_i be the number of customers served during $[t_0, t_1)$. Now serve the remaining j customers and all later arrivals until, at t_2, the system is empty. Again, $[t_1, t_2)$ is a j-busy period, and we let N_j be the number of customers served during $[t_1, t_2)$. By the independence of interarrival times and service times as well as the assumption of Poisson arrivals, the realizations of the i-busy period and the j-busy period are independent. In particular, N_i and N_j are independent, and

$$N_{i+j} = N_i + N_j \tag{1}$$

where N_{i+j} is the number of customers served during the entire $(i+j)$-busy period.

$\boxed{b}$ Extending the arguments behind (1), a k-busy period may be decomposed into k independent 1-busy periods with associated number of services $N_1(\nu)$, $\nu = 1,2,\ldots,k$, so that $N_k = \sum_{\nu=1}^{k} N_1(\nu)$. Hence,

$$E(N_k) = \sum_{\nu=1}^{k} E(N_1(\nu)) = k\,E(N_1).$$

$E(N_1)$ is most easily derived from the mean busy period b as follows. The mean cycle time equals $b+\lambda^{-1}$. Thus, the average number of busy periods (or cycles) per unit time is $[b+\lambda^{-1}]^{-1}$. Since the number of arrivals per unit time (= number of services per unit time if $\rho<1$) is λ, the average number of services per busy period will be $E(N_1) = \lambda/[b+\lambda^{-1}]^{-1} = 1+\lambda b$. According to (8.69), $b = \tau/(1-\lambda\tau)$. Hence, $E(N_1) = 1/(1-\lambda\tau) = 1/(1-\rho)$. It follows that, for $\rho<1$,

$$E(N_k) = \frac{k}{1-\rho}.$$

(Chap. 5, Ex. 17c)

$\boxed{c}$ Let $f_n^{(i)}$ denote the probability that $n \geq i$ customers will be served during an i-busy period. By part a,

$$\sum_{k=i}^{n-j} f_k^{(i)} f_{n-k}^{(j)} = f_n^{(i+j)} \qquad \left(\begin{smallmatrix} i \geq 1,\, j \geq 1, \\ n \geq i+j \end{smallmatrix}\right).$$

For an M/D/1 queue $f_n^{(i)}$ is given by (8.72), whose substitution into the above equation gives

$$\sum_{k=i}^{n-j} \frac{i}{k} \frac{(\lambda\tau k)^{k-i}}{(k-i)!} \frac{j}{n-k} \frac{(\lambda\tau(n-k))^{n-k-j}}{(n-k-j)!} = \frac{(i+j)}{n} \frac{(\lambda\tau n)^{n-i-j}}{(n-i-j)!}.$$

After cancellation of powers of $\lambda\tau$, and a rearrangement, we have

$$\sum_{k=i}^{n-j} \frac{k^{k-i-1}}{(k-i)!} \frac{(n-k)^{n-k-j-1}}{(n-k-j)!} = \frac{(i+j)}{ij} \frac{n^{n-i-j-1}}{(n-i-j)!} \qquad \left(\begin{smallmatrix} i \geq 1,\, j \geq 1, \\ n \geq i+j \end{smallmatrix}\right). \qquad (2)$$

For $i = j = 1$, (2) becomes the identity

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)\, n^{n-2} \qquad (n \geq 2). \qquad (3)$$

Chapter 5, Exercise 18

'Let N_j be the number of customers served during a j-busy period'

In the present case, the M/M/1 queue, $H^{*n}(\xi) = \sum_{k=n}^{\infty} \frac{(\mu\xi)^k}{k!} e^{-\mu\xi}$, so that

$$\frac{dH^{*n}(\xi)}{d\xi} = \frac{\mu^n}{(n-1)!} \xi^{n-1} e^{-\mu\xi}.$$

Setting $t = \infty$ in (8.73) and replacing $dH^{*n}(\xi)$ by $\frac{dH^{*n}(\xi)}{d\xi}d\xi$, we find

$$P\{N_j = n\} = \frac{j}{n}\int_0^\infty \frac{(\lambda\xi)^{n-j}}{(n-j)!} e^{-\lambda\xi} \frac{\mu^n}{(n-1)!} \xi^{n-1} e^{-\mu\xi} d\xi = \frac{j\lambda^{n-j}\mu^n}{n(n-j)!(n-1)!}\int_0^\infty \xi^{2n-j-1} e^{-(\lambda+\mu)\xi} d\xi.$$

Since $\int_0^\infty x^m e^{-ax} dx = m!/a^{m+1}$ for $a > 0$, $m = 0, 1, 2, \ldots$ (see a Table of Integrals),

$$\int_0^\infty \xi^{2n-j-1} e^{-(\lambda+\mu)\xi} d\xi = \frac{(2n-j-1)!}{(\lambda+\mu)^{2n-j}},$$

so that

$$P\{N_j = n\} = \frac{j}{n}\binom{2n-j-1}{n-j} \frac{\rho^{n-j}}{(1+\rho)^{2n-j}} \qquad (n \geq j),$$

where $\rho = \lambda/\mu$. $\square$

Chapter 5, Exercise 19

'The "polite" customer.'

Throughout in parts a–d an M/G/1 queue is assumed. By a polite customer is meant a customer who declines to enter service when any other customer is present in the queue. His equilibrium waiting time is denoted by W_p.

$\boxed{a}$ By definition, a polite customer who arrives while the server is busy will not enter service until the very end of the busy period that would have been realized without the appearance of the polite customer.

Given Poisson arrivals for all customers, we assume that also the polite customer will arrive at a random time in equilibrium. In particular, in case he arrives while the server is busy, then the arrival takes place at a randomly selected point in time in the renewal process of busy periods, disregarding the idle periods. Accordingly, the waiting time is a residual busy period, whose distribution in equilibrium is given by Eq. (7.9). It follows that

$$P\{W_p \leq x \mid W_p > 0\} = \frac{1}{b}\int_0^x [1-B(\xi)]\,d\xi, \tag{1}$$

where $B(t)$ is the distribution function of the busy period, and b is its mean.

$\boxed{b}$ With probability $1-\rho$ the polite customer arrives at an idle server and has waiting time 0. With probability ρ he arrives at a busy server and his conditional waiting time distribution is given by (1). Using that $b = \tau/(1-\rho)$, by Eq. (8.69), we conclude that

$$P\{W_p \leq x\} = (1-\rho) + \frac{\rho(1-\rho)}{\tau}\int_0^x [1-B(\xi)]\,d\xi. \tag{2}$$

$\boxed{c}$ If the polite customer has to wait, his waiting time distribution is, by part a, identical to the residual busy period distribution in equilibrium, and so the mean wait equals the mean of the residual busy period given by (7.13). Thus,

(Chap. 5, Ex. 19c)

$$E(W_p \mid W_p > 0) = \frac{b}{2} + \frac{\sigma_B^2}{2b},$$

where σ_B^2 is the variance of the busy period. As $b = \tau/(1-\rho)$,

$$E(W_p \mid W_p > 0) = \frac{\tau}{2(1-\rho)} + \frac{(1-\rho)\sigma_B^2}{2\tau}. \qquad (3)$$

$\boxed{d}$ By (8.70), $E(B^2) = \eta''(0)/(1-\rho)^3$. Now, $\eta''(0)$ equals the second moment of the service time distribution, so we may write $\eta''(0) = \sigma^2 + \tau^2$, where σ^2 is the variance of the service time. Hence,

$$\sigma_B^2 = E(B^2) - b^2 = \frac{\sigma^2+\tau^2}{(1-\rho)^3} - \frac{\tau^2}{(1-\rho)^2}$$

$$= \frac{\sigma^2+\rho\tau^2}{(1-\rho)^3}.$$

Substitution of this expression for σ_B^2 into (3) yields

$$E(W_p \mid W_p > 0) = \frac{\tau^2+\sigma^2}{2\tau(1-\rho)^2}. \qquad (4)$$

$\boxed{e}$ Suppose the polite customer makes his arrival in an M/M/s queue, with arrival rate λ and mean service time μ^{-1}. There he will wait only if all s servers are busy on arrival.

The key observation is that all-servers-busy periods follow precisely the same distribution as does the busy period in the M/M/1 queue with arrival rate λ and mean service time $(s\mu)^{-1}$.

We can conclude that $E(W_p \mid W_p > 0)$ may be derived by the use of Eq. (4), setting $\tau = (s\mu)^{-1}$, $\sigma^2 = \tau^2 = (s\mu)^{-2}$, and $\rho = \lambda/s\mu$. Hence, for the M/M/s queue,

$$E(W_p \mid W_p > 0) = \frac{(s\mu)^{-2} + (s\mu)^{-2}}{2(s\mu)^{-1}(1-\rho)^2},$$

simplifying to

$$E(W_p \mid W_p > 0) = \frac{1}{(1-\rho)^2 s\mu}. \qquad (5)$$

□

Chapter 5, Exercise 20

'Service in reverse order of arrival'

$\boxed{a}$ We consider an arbitrary customer arriving at an M/G/1 queue at a time T_c when the server is busy. Disregarding idle periods, the arrival epoch is a randomly selected point in time in the renewal process where interevent times have probability distribution $H(t)$. Hence, the remaining service time T_1-T_c has the probability distribution function $\tilde{H}(t)$ given by (8.41), by application of (7.9).

If $T_1-T_c=t$, then the number of new arrivals during $[T_c,T_1)$ will follow the Poisson distribution with mean λt. Thus, the joint probability of $T_1-T_c\leq x$ and j new arrivals is

$$\tilde{P}_j(x)=\int_0^x \frac{(\lambda\xi)^j}{j!}e^{-\lambda\xi}d\,\tilde{H}(\xi).$$

$\boxed{b}$ Let, as usual, W be the waiting time of an arbitrary customer (the test customer) and let $W(t)$ be the equilibrium waiting time distribution function, given service in reverse order of arrival. Denoting the arrival state by N, clearly

$$W(t)=P\{W\leq t\}=P\{N=0\}P\{W\leq t|N=0\}+P\{N\geq 1\}P\{W\leq t|N\geq 1\}.$$

Given Poisson arrivals, $P\{N\geq 1\}=\rho$. Hence,

$$W(t)=(1-\rho)+\rho\,P\{W\leq t|N\geq 1\}.$$

To find $P\{W\leq t|N\geq 1\}$, observe that the waiting time is the sum of the remaining service time T_1-T_c and a j-busy period, where j is the number of arrivals during $[T_c,T_1)$, since the test customer must wait until both these j customers and all later arrivals have been served.

Note that for given j, the conditional remaining service time and the j-busy period are independent. It follows that, given $N\geq 1$, the joint probability of j new arrivals and a total wait of less than t equals $\int_0^t \tilde{P}_j(t-x)\,d\,B_j(x)$, where $B_j(x)$ is the distribution function of the j-busy period. Hence,

$$P\{W\leq t|N\geq 1\}=\sum_{j=0}^{\infty}\int_0^t \tilde{P}_j(t-x)\,d\,B_j(x),$$

(Chap. 5, Ex. 20 b)

so that

$$W(t) = (1-\rho) + \rho \sum_{j=0}^{\infty} \int_0^t \tilde{P}_j(t-x)\, d B_j(x). \qquad (1)$$

c Let $\omega(s)$ denote the Laplace-Stieltjes transform of the waiting time distribution function $W(t)$, and let, as before, $\eta(s)$ and $\beta(s)$ be the Laplace-Stieltjes transforms of service time and busy period distribution functions, respectively.

By (1), and part a,

$$\begin{aligned}
\omega(s) &= \int_0^\infty e^{-st} d W(t) \\
&= (1-\rho) + \rho \sum_{j=0}^{\infty} \left(\int_0^\infty e^{-sx} d \tilde{P}_j(x) \right) \left(\int_0^\infty e^{-sx} d B_j(x) \right) \\
&= (1-\rho) + \rho \sum_{j=0}^{\infty} \left(\int_0^\infty e^{-sx} \frac{(\lambda x)^j}{j!} e^{-\lambda x} d \tilde{H}(x) \right) \beta^j(s) \\
&= (1-\rho) + \rho \sum_{j=0}^{\infty} \int_0^\infty e^{-(s+\lambda)x} \frac{(\lambda x \beta(s))^j}{j!} d \tilde{H}(x) \\
&= (1-\rho) + \rho \int_0^\infty e^{-(s+\lambda)x} \left(\sum_{j=0}^{\infty} \frac{(\lambda x \beta(s))^j}{j!} \right) d \tilde{H}(x) \\
&= (1-\rho) + \rho \int_0^\infty e^{-(s+\lambda[1-\beta(s)])x} d \tilde{H}(x).
\end{aligned}$$

The last integral is the Laplace-Stieltjes transform of the remaining service time distribution function, evaluated at $s+\lambda[1-\beta(s)]$. According to Eq. (7.10) this transform equals

$$\int_0^\infty e^{-sx} d \tilde{H}(x) = \frac{1}{\tau} \frac{1-\eta(s)}{s},$$

where τ is the mean service time. Hence, since $\rho = \lambda\tau$,

$$\omega(s) = (1-\rho) + \lambda \frac{1-\eta(s+\lambda-\lambda\beta(s))}{s+\lambda[1-\beta(s)]}. \qquad (2)$$

d By (8.67), $\beta(s) = \eta(s+\lambda-\lambda\beta(s))$, so that (2) reduces to

$$\omega(s) = (1-\rho) + \frac{\lambda[1-\beta(s)]}{s+\lambda[1-\beta(s)]}. \qquad (3)$$

□

Chapter 5, Exercise 21

'It is required to calculate the arriving customer's equilibrium distribution $\{\Pi_j\}$ for the M/G/1 queue with n waiting positions.'

$\boxed{a}$ Let $\tilde{\Pi}_j^*$ ($j = 0,1,\dots,n$) denote any unnormalized Π_j^* calculated from (9.1) starting with an arbitrary positive value of Π_0^*, and set

$$d = \tilde{\Pi}_0^* + \tilde{\Pi}_1^* + \dots + \tilde{\Pi}_n^* .$$

By (9.12) and (9.13),

$$\Pi_j = \frac{\Pi_j^*}{\Pi_0^* + a} = \frac{\Pi_j^* d}{\Pi_0^* d + ad} = \frac{\tilde{\Pi}_j^*}{\tilde{\Pi}_0^* + ad} \qquad (j = 0,1,\dots,n),$$

$$\Pi_{n+1} = \frac{\Pi_0^* + (a-1)}{\Pi_0^* + a} = \frac{\Pi_0^* d + (a-1)d}{\Pi_0^* d + ad} = \frac{\tilde{\Pi}_0^* + (a-1)d}{\tilde{\Pi}_0^* + ad} .$$

$\boxed{b}$ Suppose that the state distribution $\{\hat{\Pi}_j\}$ for the corresponding infinite-waiting-room queue has been calculated.

By proportionality of $\{\hat{\Pi}_j\}$ and $\{\Pi_j^*\}$ for $j = 0,1,\dots,n$, the starting value $\tilde{\Pi}_0^* = \hat{\Pi}_0$ will lead to $\tilde{\Pi}_j^* = \hat{\Pi}_j$ ($j = 0,1,\dots,n$) and $d = \Sigma_{\nu=0}^{n} \hat{\Pi}_\nu$. By part a, then,

$$\Pi_j = \frac{\hat{\Pi}_j}{\hat{\Pi}_0 + a\Sigma_{\nu=0}^{n}\hat{\Pi}_\nu} \qquad (j = 0,1,\dots,n),$$

$$\Pi_{n+1} = \frac{\hat{\Pi}_0 + (a-1)\Sigma_{\nu=0}^{n}\hat{\Pi}_\nu}{\hat{\Pi}_0 + a\Sigma_{\nu=0}^{n}\hat{\Pi}_\nu} .$$

$\boxed{c}$ For an M/M/1 queue $\hat{\Pi}_j = (1-a)a^j$ ($j = 0,1,\dots$), see Ex. 4 of Chapter 1. By substitution into the equations of part b:

$$\Pi_j = \frac{(1-a)a^j}{1-a^{n+2}} \qquad (j = 0,1,\dots,n+1).$$

The "rate up = rate down" equations are $\lambda P_j = \mu P_{j+1}$ ($j = 0,1,\dots,n$), by which $\Pi_j = P_j = (1-a)a^j/(1-a^{n+2})$, where $a = \lambda/\mu$, in agreement with the above result. $\square$

Chapter 5, Exercise 22

'A particle-counting device...'

The system may be modeled as a single-server queue with waiting room of size 2, gross arrival rate 3λ, effective arrival rate in state j equal to $\lambda_j = (3-j)\lambda$, constant service time $\tau = 1$. Let

b_j = mean of a j-busy period ($j = 1,2$),

$p(i|j)$ = probability that i buffers fill (i particles arrive at idle buffers) during a service time, given state j ($j = 1,2$) at the start of service.

Clearly,

$$b_1 = 1 + p(1|1)b_1 + p(2|1)b_2, \tag{1}$$
$$b_2 = 1 + p(0|2)b_1 + p(1|2)b_2. \tag{2}$$

Using $p(0|2) + p(1|2) = 1$, Eq. (2) can be written

$$b_2 = \frac{1 + p(0|2)\,b_1}{p(0|2)}. \tag{3}$$

Substitution of (3) into (1) and use of $p(0|1) + p(1|1) + p(2|1) = 1$ give

$$b_1 = \frac{p(0|2) + p(2|1)}{p(0|1)\,p(0|2)}. \tag{4}$$

The probability that an idle buffer will be filled during a service period equals $1 - e^{-\lambda}$. Hence, $p(0|2) = e^{-\lambda}$, $p(2|1) = (1 - e^{-\lambda})^2$, and $p(0|1) = (e^{-\lambda})^2$. Substitution into (4) and simplification yield

$$b_1 = e^{3\lambda} + e^{\lambda} - e^{2\lambda} \tag{5}$$

As $\lambda_0 = 3\lambda$, the mean idle period equals $(3\lambda)^{-1}$. Hence, in analogy with (9.14), the carried load is given by

$$a' = \frac{b_1}{(3\lambda)^{-1} + b_1}. \tag{6}$$

The offered load is

$$a = (3\lambda)\tau = 3\lambda. \tag{7}$$

By (5), (6) and (7),

$$p = 1 - \frac{a'}{a} = 1 - \frac{e^{3\lambda} + e^{\lambda} - e^{2\lambda}}{1 + 3\lambda(e^{3\lambda} + e^{\lambda} - e^{2\lambda})}. \qquad \square$$

Chapter 5, Exercise 23

'Let $B(j,k)$ be the duration of the j-busy period in the M/G/1 queue with $j+k-1$ waiting positions ...'

Observe that the system may hold altogether $j+k$ customers. Let $E[B(j,k)] = b(j,k)$, and let $P(j,k)$ be the probability that throughout the busy period there will always be at least one unoccupied waiting position.

$\boxed{a}$ Suppose service begins when there are j customers in the system. We may assume that customers are served in reverse order-of-arrival. Initially, we decompose the j-busy period into two independent time intervals. The first interval is the time needed to reduce the state from j to $j-1$. This time interval is distributed as $B(1,k)$. The second interval is the time needed to reduce the state from $j-1$ to 0, so this time interval is distributed as $B(j-1,k+1)$ assuming $j \geq 2$. Hence,

$$B(j,k) = B(1,k) + B(j-1,k+1) \quad (j \geq 2), \tag{1}$$

so that

$$B(j,k) = \sum_{i=k}^{j+k-1} B(1,i). \tag{2}$$

$\boxed{b}$ Taking means in (2) we obtain

$$b(j,k) = \sum_{i=k}^{j+k-1} b(1,i). \tag{3}$$

We now decompose the j-busy period in a different way. Imagine that the first customer, C, if any, who fills up the queue will not enter service until there are no other waiting customers. Then the time until C, should he exist, will get served is distributed as $B(j,k-1)$, assuming $k \geq 1$. With probability $1-P(j,k)$, C will arrive during the busy period and thus generate a 1-busy period distributed as $B(1,j+k-1)$.

We conclude that

$$B(j,k) = B(j,k-1) + I * B(1,j+k-1) \quad (k \geq 1), \tag{4}$$

(Chap. 5, Ex. 23 b)

where $I = 0$ if C does not arrive and $I = 1$ if he does. Note that I and $B(1, j+k-1)$ are independent variables. Taking means in (4) we derive

$$b(j,k) = b(j,k-1) + [1 - P(j,k)]\, b(1, j+k-1) \qquad (k \geq 1). \qquad (5)$$

$\boxed{c}$ Writing $b(j,k)$ and $b(j,k-1)$ as sums of mean 1-busy periods, by the use of (3), Eq. (5) yields

$$P(j,k) = \frac{b(1,k-1)}{b(1,j+k-1)} \qquad (k \geq 1). \qquad (6)$$

$\boxed{d}$ Now assume exponential service times with mean μ^{-1}, and let $a = \lambda/\mu$. In this case the mean busy periods are:

$$M/M/1/n: \qquad b(1,n) = \frac{1}{\mu}\sum_{i=0}^{n} a^i \qquad (n \geq 0). \qquad (7)$$

This formula might be derived from (9.18). Alternatively, one can use the relation $b(1,n)/\lambda^{-1} = (1-P_0)/P_0$ (compare with Eq. (4.10) of Chapter 3) plus the fact that $P_0 = 1/\sum_{i=0}^{n+1} a^i$. By (6) and (7),

$$M/M/1/j+k-1: \qquad P(j,k) = \frac{\sum_{i=0}^{k-1} a^i}{\sum_{i=0}^{j+k-1} a^i} \qquad (k \geq 1). \qquad (8)$$

$\boxed{e}$ Define $P(j,k,a) = P(j,k)$, and let $P'(j,k,p)$ denote the gamblers ruin probability in a game where he starts with j units, the adversary starts with k units, and the probability of his winning 1 unit is p in each trial (thus the adversary's winning probability is $q = 1-p$). We shall show that

$$P'(j,k,p) = P(j,k,p/q). \qquad (9)$$

Consider an M/M/1 queue with $j+k$ positions (incl. service) where $j \geq 1$, $k \geq 1$, and j is the initial state. In state i, $1 \leq i \leq j+k-1$, the transition probabilities are $P\{i \to i+1\} = \lambda/(\lambda+\mu) = a/(1+a) = p$, $P\{i \to i-1\} = 1-p = q$. One sees that the imbedded Markov chain (arrivals and departures) of the state variable i, exactly simulates the game as described if λ and μ satisfy $\lambda/(\lambda+\mu) = p$. Eq. (9) follows. □

Chapter 5, Exercise 24

'Consider the equilibrium M/G/1 queue with batch arrivals...'

W_1 = time from arrival to start of service of the test customer's batch
W_2 = remaining time until start of service of the test customer

Let $\omega_1(s)$ and $\omega_2(s)$ denote the Laplace-Stieltjes transforms of W_1 and W_2, respectively. As W_1 and W_2 are independent, $W = W_1 + W_2$ has Laplace-Stieltjes transform

$$\hat{\omega}(s) = \omega_1(s)\,\omega_2(s) \tag{1}$$

Derivation of $\omega_1(s)$

Let τ_B be the mean and let $\eta_B(s)$ be the Laplace-Stieltjes transform of the batch service time. By (8.38),

$$\omega_1(s) = \frac{s(1-\lambda\tau_B)}{s-\lambda[1-\eta_B(s)]}\,.$$

By Exercise 5, $\tau_B = m\tau$ and $\eta_B(s) = g(\eta(s))$, where τ and $\eta(s)$ are mean and Laplace-Stieltjes transform of the individual service times, and m and $g(z)$ are mean and probability generating function of the number of customers in a batch.
Thus,

$$\omega_1(s) = \frac{s(1-\lambda m\tau)}{s-\lambda[1-g(\eta(s))]}\,. \tag{2}$$

Derivation of $\omega_2(s)$

Let N and N' be, respectively, the size of an arbitrary batch and a test customer's batch, and let N'' be the number of batch customers served ahead of the test customer. We may assume that the customers in a batch are served in random order, but batches must be served in order of arrival.
Clearly,

$$P\{N''=k\} = \sum_{j=k+1}^{\infty} \frac{1}{j} P\{N'=j\} \qquad (k=0,1,\ldots).$$

Substitution therein of $P\{N'=j\} = j\,P\{N=j\}/m$, by (10.6), results in

(Chap. 5, Ex. 24)

$$P\{N''=k\} = \frac{1}{m}\sum_{j=k+1}^{\infty} P\{N=j\} \qquad (k=0,1,\ldots).$$

Denoting by $h(z)$ the probability generating function of N'', we find

$$\begin{aligned} h(z) &= \sum_{k=0}^{\infty} P\{N''=k\} z^k \\ &= \sum_{k=0}^{\infty} \frac{1}{m} \sum_{j=k+1}^{\infty} P\{N=j\} z^k \\ &= \frac{1}{m}\sum_{k=0}^{\infty} z^k - \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{k} P\{N=j\} z^k \\ &= \frac{1}{m}\sum_{k=0}^{\infty} z^k - \frac{1}{m}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty} P\{N=j\} z^k \\ &= \frac{1}{m}\sum_{k=0}^{\infty} z^k - \frac{1}{m}\sum_{j=0}^{\infty} P\{N=j\} z^j \sum_{k=0}^{\infty} z^k \\ &= \frac{1}{m}\Big[1-\sum_{j=0}^{\infty} P\{N=j\} z^j\Big]\sum_{k=0}^{\infty} z^k, \end{aligned}$$

whereby

$$h(z) = \frac{1-g(z)}{m[1-z]}. \tag{3}$$

Now, W_2 is the sum of N'' independent service times, each with Laplace-Stieltjes transform $\eta(s)$. By Exercise 5,

$$\omega_2(s) = h(\eta(s)). \tag{4}$$

By (3) and (4),

$$\omega_2(s) = \frac{1-g(\eta(s))}{m[1-\eta(s)]}. \tag{5}$$

Finally, combining (1), (2) and (5), we obtain the Laplace-Stieltjes transform of an arbitrary customer's total waiting time W,

$$\hat{\omega}(s) = \frac{s(1-\lambda m\tau)}{s-\lambda[1-g(\eta(s))]}\,\frac{1-g(\eta(s))}{m[1-\eta(s)]}.$$

$\square$

Chapter 5, Exercise 25

'Let c be the minimum mean operating cost per unit time ...'

We assume an M/G/1 queue and the choice between continuous operation and some N-policy. As a consequence of the preceding analysis we distinguish between two cases as follows.

Case 1: $c_0 < \lambda c_N(n^*)/(1-\rho)$

It is known already that in this case c is minimized by continuous operation. Per unit time, the three cost elements are

$$\begin{aligned}\text{running cost} &= c_0,\\ \text{switching cost} &= 0,\\ \text{holding cost} &= \lambda c_2 E(X).\end{aligned}$$

Now, $E(X) = \tau + E(W)$, and the mean waiting time $E(W)$ is given by the Pollaczek-Khintchine formula (8.39). Thus $c = c_0 + \lambda c_2 E(X)$ becomes

$$c = c_0 + c_2\left[\rho + \frac{\rho^2}{2(1-\rho)}\left(1+\frac{\sigma^2}{\tau^2}\right)\right] \qquad \left(c_0 < \frac{\lambda c_N(n^*)}{1-\rho}\right). \qquad (*)$$

Case 2: $c_0 > \lambda c_N(n^*)/(1-\rho)$

In this case c is minimized by choosing an N-policy with parameter $n = n^*$. The calculation of the corresponding cost $c_N(n^*)$, namely the minimal variable cost per customer, has been described previously. The minimal variable cost per unit time is seen to equal $\lambda c_N(n^*)$. c is obtained by adding those fixed costs (independent of n) that were not taken into consideration in calculating n^*.

First, there is a running cost per unit time equal to $c_0\rho$, since, for any n, the server will be busy (= on) the fraction ρ of the time. Second, there is the holding cost $\lambda c_2 E(X)$ of a system where the server is on whenever a customer is present. Thus, $c = c_0\rho + \lambda c_N(n^*) + \lambda c_2 E(X)$, which becomes

$$c = c_0\rho + \lambda c_N(n^*) + c_2\left[\rho + \frac{\rho^2}{2(1-\rho)}\left(1+\frac{\sigma^2}{\tau^2}\right)\right] \quad \left(c_0 > \frac{\lambda c_N(n^*)}{1-\rho}\right). \quad (**)$$

(Chap. 5, Ex. 25)

Examples

In every case, $\lambda = \frac{1}{2}$, $\tau = 1$ (so that $\rho = \frac{1}{2}$), $c_1 = 12$ and $c_2 = 2$. Hence, $n^* = 2$, $c_N(n^*) = 5$, and the borderline value for c_0 is $\hat{c}_0 = \lambda c_N(n^*)/(1-\rho) = 5$.

(a) $c_0 = 4$. As $c_0 < \hat{c}_0$, c is minimized by a do-nothing policy. Thus, Eq. (*) applies. If service times are exponentially distributed, then $\sigma^2 = \tau^2 = 1$ and, by (*), $c = 6$. If service times are constant, then $\sigma^2 = 0$ and, by (*), $c = 5\frac{1}{2}$.

(b) $c_0 = 6$. As $c_0 > \hat{c}_0$, c is minimized by an N-policy with $n^* = 2$. Thus, Eq. (**) applies. If service times are exponentially distributed, then $\sigma^2 = \tau^2 = 1$ and, by (**), $c = 7\frac{1}{2}$. If service times are constant, then $\sigma^2 = 0$ and, by (**), $c = 7$.

Chapter 5, Exercise 26

'Consider the M/G/1 queue operating under a T-policy, with parameter t'

a The probability that no customer will arrive during a vacation of length t is $P(0) = e^{-\lambda t}$. Hence, with Y being the consecutive number of vacations with no arrivals, $P\{Y = i\} = (e^{-\lambda t})^n(1-e^{-\lambda t})$, whereby

$$E(Y) = \sum_{i=0}^{\infty} i\, P\{Y = i\} = \frac{e^{-\lambda t}}{1 - e^{-\lambda t}}.$$

b Given Poisson traffic, the number of arrivals during a vacation has the Poisson distribution with mean λt. Hence,

$$f_T(z) = \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} z^j = e^{-(1-z)\lambda t}.$$

c
$$f_T'(z) = \lambda t e^{-(1-z)\lambda t}, \quad f_T''(z) = \lambda^2 t^2 e^{-(1-z)\lambda t},$$
$$f_T'(1) = \lambda t \quad , \quad f_T''(1) = \lambda^2 t^2.$$

If in (11.14) we substitute $P(0) = e^{-\lambda t}$ and the above expressions for $E(Y)$, $f_T'(1)$ and $f_T''(1)$, we obtain (11.24). □

$\boxed{\text{Chapter 5, Exercise 27}}$

'The Maclaurin series method for the M/G/1 random service queue.'

$$F(t) \equiv P\{W>t \mid W>0\} = 1 - \tilde{H}(t) + \sum_{j=2}^{\infty} \frac{j-1}{j} \int_0^t \breve{W}_j(t-\xi) Q_j'(\xi)\, d\xi . \tag{12.9}$$

$$\breve{W}_j(x) = 1 - H(x) + \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1} \int_0^x p_i(\xi) \breve{W}_{j+i-1}(x-\xi)\, dH(\xi) \quad (j=2,3,\ldots). \tag{12.10}$$

Assume that $\breve{W}_j(x)$ has the Maclaurin series expansion

$$\breve{W}_j(x) = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} \breve{W}_j^{(\nu)} \quad (j = 2,3,\ldots;\ \breve{W}_j^{(0)} = 1). \tag{1}$$

Suppose $H(x)$ is continuous and differentiable, and set

$$h(x) = \frac{d}{dx} H(x), \tag{2}$$

and let

$$\breve{b}_i(x) = h(x)\, p_i(x). \tag{3}$$

Also, for any function $f(x)$, define $f^{(k)} = \left(\frac{d^k}{dx^k} f(x)\right)_{x=0}$.

$\boxed{a}$ By (12.10), (2) and (3),

$$\breve{W}_j(x) = 1 - H(x) + \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1} \int_0^x \breve{b}_i^{(0)}(\xi) \breve{W}_{j+i-1}^{(0)}(x-\xi)\, d\xi \quad (j = 2,3,\ldots).$$

Repeated differentiation w.r.t. x yields

$$\breve{W}_j^{(\nu)}(x) = -H^{(\nu)}(x) + \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1} \left[\sum_{k=0}^{\nu-1} \breve{b}_i^{(k)}(x) \breve{W}_{j+i-1}^{(\nu-1-k)}(0) + \int_0^x \breve{b}_i^{(0)}(\xi) \breve{W}_{j+i-1}^{(\nu)}(x-\xi) d\xi \right]$$

for $j = 2,3,\ldots;\ \nu = 1,2,\ldots$.
Hence,

$$\breve{W}_j^{(\nu)} = -H^{(\nu)} + \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1} \sum_{k=0}^{\nu-1} \breve{b}_i^{(k)} \breve{W}_{j+i-1}^{(\nu-1-k)} \quad \left(\begin{smallmatrix} j=2,3,\ldots; \\ \nu=1,2,\ldots \end{smallmatrix}\right).$$

It is easy to show that $p_i^{(k)} = 0$ for $i > k$. Hence, $\breve{b}_i^{(k)} = 0$ for $i > k$, so that the above equation simplifies to

$$\breve{W}_j^{(\nu)} = -H^{(\nu)} + \sum_{i=0}^{\nu-1} \frac{j+i-2}{j+i-1} \sum_{k=i}^{\nu-1} \breve{b}_i^{(k)} \breve{W}_{j+i-1}^{(\nu-1-k)} \quad \left(\begin{smallmatrix} j=2,3,\ldots; \\ \nu=1,2,\ldots \end{smallmatrix}\right). \tag{4}$$

(Chap. 5, Ex. 27 b)

$\boxed{b}$ Defining $F(t) = P\{W>t \mid W>0\}$ we assume the Maclaurin series expansion

$$F(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} F^{(\nu)} \qquad (F^{(0)} = 1). \tag{5}$$

Using our present notation, (12.4) is written

$$F(t) = 1 - \tilde{H}(t) + \sum_{j=2}^{\infty} \frac{j-1}{j} \int_0^t \breve{W}_j^{(0)}(t-\xi) Q_j^{(1)}(\xi)\, d\xi .$$

Repeated differentiation w.r.t. t yields

$$F^{(\nu)}(t) = -\tilde{H}^{(\nu)}(t) + \sum_{j=2}^{\infty} \frac{j-1}{j} \left[\sum_{k=0}^{\nu-1} \breve{W}_j^{(\nu-1-k)}(0) Q_j^{(k+1)}(t) + \int_0^t \breve{W}_j^{(\nu)}(t-\xi) Q_j^{(1)}(\xi)\, d\xi \right]$$

for $\nu = 1, 2, \ldots$. Hence,

$$F^{(\nu)} = -\tilde{H}^{(\nu)} + \sum_{j=2}^{\infty} \frac{j-1}{j} \sum_{k=0}^{\nu-1} Q_j^{(k+1)} \breve{W}_j^{(\nu-1-k)} \quad (\nu = 1, 2, \ldots). \tag{6}$$

$\boxed{c}$ $$Q_j^{(1)}(\xi) = \frac{1}{\tau} \int_\xi^\infty \left(\Pi_0^* p_{j-1}(x) + \sum_{i=1}^{j} \Pi_i^* p_{j-i}(x) \right) dH(x) \quad (j = 1, 2, \ldots). \tag{12.7}$$

Hence,

$$Q_j^{(1)} = \frac{1}{\tau} \int_0^\infty \left(\Pi_0^* p_{j-1}(x) + \sum_{i=1}^{j} \Pi_i^* p_{j-i}(x) \right) dH(x) \quad (j = 1, 2, \ldots).$$

The integrand (in parentheses) is the equilibrium probability of departure state $j-1$, given service time x. Thus, the integration results in the unconditional probability of departure state $j-1$, that is, Π_{j-1}^*. We conclude that

$$Q_j^{(1)} = \frac{1}{\tau} \Pi_{j-1}^* \quad (j = 1, 2, \ldots). \tag{7}$$

Repeated differentiation of (12.7) w.r.t. ξ yields

$$Q_j^{(k+1)}(\xi) = -\frac{1}{\tau} \sum_{m=0}^{k-1} \binom{k-1}{m} H^{(k-m)}(\xi) \left(\Pi_0^* p_{j-1}^{(m)}(\xi) + \sum_{i=1}^{j} \Pi_i^* p_{j-i}^{(m)}(\xi) \right) \quad \binom{k = 1, 2, \ldots;}{j = 1, 2, \ldots},$$

whereby

$$Q_j^{(k+1)} = -\frac{1}{\tau} \sum_{m=0}^{k-1} \binom{k-1}{m} H^{(k-m)} \left(\Pi_0^* p_{j-1}^{(m)} + \sum_{i=1}^{j} \Pi_i^* p_{j-i}^{(m)} \right) \quad \binom{k = 1, 2, \ldots;}{j = 1, 2, \ldots} \tag{8}$$

□

Chapter 5, Exercise 28

(Carter and Cooper [1972]) – cf. Ex. 32 of Chap. 3

The exercise is an application of the results of Exercise 27 on the M/M/1 random service queue. Thus we assume

$$H(x) = 1 - e^{-x/\tau}. \qquad (1)$$

Clearly, $\tilde{H}(x) = H(x) = 1 - e^{-x/\tau}$, whereby

$$\tilde{H}^{(1)}(x) = \frac{1}{\tau} e^{-x/\tau}, \qquad (*)$$

$$\tilde{H}^{(2)}(x) = -\frac{1}{\tau^2} e^{-x/\tau}. \qquad (**)$$

$\boxed{a}$ By Eq. (6) of Exercise 27,

$$F^{(1)} = -\tilde{H}^{(1)} + \sum_{j=2}^{\infty} \frac{j-1}{j} Q_j^{(1)} \breve{W}_j^{(0)}.$$

By (*) and the fact that $\breve{W}_j^{(0)} = \breve{W}_j(0) = 1$ for $j \geq 2$, then,

$$F^{(1)} = -\frac{1}{\tau} + \sum_{j=2}^{\infty} \frac{j-1}{j} Q_j^{(1)}. \qquad (2)$$

$\boxed{b}$ By Eq. (7) of Exercise 27, $Q_j^{(1)} = \frac{1}{\tau} \Pi_{j-1}^*$ $(j = 1, 2, \ldots)$. For an M/M/1 queue, $\Pi_j^* = P_j = (1-\rho)\rho^j$, where $\rho = \lambda\tau$. Thus

$$Q_j^{(1)} = \frac{1}{\tau}(1-\rho)\rho^{j-1} \qquad (j = 1, 2, \ldots). \qquad (***)$$

By (2) and (***),

$$F^{(1)} = -\frac{1}{\tau} + \frac{1}{\tau}\frac{1-\rho}{\rho} \sum_{j=1}^{\infty} \frac{j-1}{j} \rho^j$$

$$= -\frac{1}{\tau} + \frac{1}{\tau}\frac{1-\rho}{\rho} \sum_{j=1}^{\infty} \rho^j - \frac{1}{\tau}\frac{1-\rho}{\rho} \sum_{j=1}^{\infty} \frac{\rho^j}{j}$$

$$= -\frac{1}{\tau}\frac{1-\rho}{\rho} \sum_{j=1}^{\infty} \frac{\rho^j}{j}.$$

Hence,

$$F^{(1)} = -\frac{1}{\tau}\frac{1-\rho}{\rho} \ln \frac{1}{1-\rho}. \qquad (3)$$

(Chap. 5, Ex. 28 c)

c By Eq. (6) of Exercise 27,

$$F^{(2)} = -\widetilde{H}^{(2)} + \sum_{j=2}^{\infty} \frac{j-1}{j}\left(Q_j^{(1)}\breve{W}_j^{(1)} + Q_j^{(2)}\breve{W}_j^{(0)}\right).$$

By (* *) and the fact that $\breve{W}_j^{(0)} = \breve{W}_j(0) = 1$ for $j \geq 2$, then,

$$F^{(2)} = \frac{1}{\tau^2} + \sum_{j=2}^{\infty} \frac{j-1}{j}\left(Q_j^{(1)}\breve{W}_j^{(1)} + Q_j^{(2)}\right) \qquad (4)$$

d By Eq. (8) of Exercise 27, where $\binom{0}{0} = 1$, $p_{j-1}^{(0)} = p_{j-1}(0)$, $p_{j-i}^{(0)} = p_{j-i}(0)$,

$$Q_j^{(2)} = -\frac{1}{\tau} H^{(1)}\left(\pi_0^* p_{j-1}(0) + \sum_{i=1}^{j} \pi_i^* p_{j-i}(0)\right) \quad (j=1,2,\ldots). \qquad (5)$$

As $p_i(x) = [(\lambda x)^i/i!]e^{-\lambda x}$, we have $p_0(0) = 1$ and $p_i(0) = 0$ for $i \geq 1$. Substitution of these values for $p_{j-1}(0)$ and $p_{j-i}(0)$ as well as $H^{(1)} = \frac{1}{\tau}$ into (5), but only for $j = 2,3,\ldots$, we obtain

$$Q_j^{(2)} = -\frac{1}{\tau^2}\pi_j^* \qquad (j = 2,3,\ldots).$$

Finally, since $\pi_j^* = P_j = (1-\rho)\rho^j$,

$$Q_j^{(2)} = -\frac{1}{\tau^2}(1-\rho)\rho^j \qquad (j = 2,3,\ldots). \qquad (6)$$

e By Eq. (4) of Exercise 27,

$$\breve{W}_j^{(1)} = -H^{(1)} + \frac{j-2}{j-1}\breve{b}_0^{(0)}\breve{W}_{j-1}^{(0)} \qquad (j = 2,3,\ldots).$$

By definition $\breve{b}_0^{(0)} = \breve{b}_0(0) = H^{(1)}(0)p_0(0)$. As $H^{(1)}(0) = H^{(1)} = \frac{1}{\tau}$ and $p_0(0) = 1$, we have $\breve{b}_0^{(0)} = \frac{1}{\tau}$. Also, $\breve{W}_1^{(0)} = \breve{W}_1(0) = 0$, and $\breve{W}_j^{(0)} = \breve{W}_j(0) = 1$ for $j \geq 2$. We conclude that

$$\breve{W}_j^{(1)} = \begin{cases} -\frac{1}{\tau} & (j = 2), \\ -\frac{1}{\tau} + \frac{j-2}{j-1}\frac{1}{\tau} & (j \geq 3), \end{cases}$$

or,

$$\breve{W}_j^{(1)} = -\frac{1}{\tau}\frac{1}{j-1} \qquad (j = 2,3,\ldots). \qquad (7)$$

(Chap. 5, Ex. 28 f)

$\boxed{f}$ Substitution into (4) of the expressions that have been derived for $Q_j^{(1)}$, $Q_j^{(2)}$ and $\tilde{W}_j^{(1)}$, and subsequent reduction, give

$$F^{(2)} = \frac{1}{\tau^2} + \sum_{j=2}^{\infty} \frac{j-1}{j}\left(\left[\frac{1}{\tau}(1-\rho)\rho^{j-1}\right]\left[-\frac{1}{\tau}\frac{1}{j-1}\right] + \left[-\frac{1}{\tau^2}(1-\rho)\rho^j\right]\right)$$

$$= \frac{1}{\tau^2}\left[1 - \frac{1-\rho}{\rho}\sum_{j=2}^{\infty}\frac{\rho^j}{j} - (1-\rho)\sum_{j=2}^{\infty}\frac{j-1}{j}\rho^j\right]$$

$$= \frac{1}{\tau^2}\left[1 - \frac{1-\rho}{\rho}\left(\sum_{j=1}^{\infty}\frac{\rho^j}{j} - \rho\right) - (1-\rho)\left(\sum_{j=1}^{\infty}\rho^j - \sum_{j=1}^{\infty}\frac{\rho^j}{j}\right)\right]$$

$$= \frac{1}{\tau^2}\left[1 - \frac{1-\rho}{\rho}\sum_{j=1}^{\infty}\frac{\rho^j}{j} + (1-\rho) - \rho + (1-\rho)\sum_{j=1}^{\infty}\frac{\rho^j}{j}\right]$$

$$= \frac{1}{\tau^2}(1-\rho)\left[2 - \frac{1-\rho}{\rho}\sum_{j=1}^{\infty}\frac{\rho^j}{j}\right].$$

Thus, as expected,

$$F^{(2)} = \frac{1}{\tau^2}(1-\rho)\left[2 - \frac{1-\rho}{\rho}\ln\frac{1}{1-\rho}\right]. \tag{8}$$

$\boxed{g}$ Also Eq. (6) of Exercise 32 of Chapter 3 expresses the conditional waiting time distribution function in terms of a Maclaurin series expansion, but for an M/M/s random service queue. For $s=1$ the formula specializes to

$$P\{W>t \mid W>0\} = 1 + tF^{(1)} + \frac{t^2}{2!}F^{(2)} + \cdots$$

where

$$F^{(1)} = -\frac{1}{\tau}\frac{1-\rho}{\rho}\ln\frac{1}{1-\rho},$$

$$F^{(2)} = \frac{1}{\tau^2}(1-\rho)\left[2 - \frac{1-\rho}{\rho}\ln\frac{1}{1-\rho}\right],$$

in complete agreement with Eqs. (3) and (8). $\square$

Chapter 5, Exercise 29

'The additional-conditioning-variable method for the M/D/1 random-service-queue (Carter and Cooper [1972].'

We assume a service time equal to the constant τ, i.e.

$$H(x) = \begin{cases} 0 & \text{when } x < \tau, \\ 1 & \text{when } x \geq \tau. \end{cases} \qquad (1)$$

$\boxed{a}$ By (1), $dH(\tau) = 1$ and $dH(x) = 0$ for $x \neq \tau$. Inserting this into Eq. (12.7) we find that for $\xi > \tau$ is $Q_j'(\xi) = 0$ whereas for $\xi \leq \tau$,

$$\begin{aligned} Q_j'(\xi) &= \frac{1}{\tau}\int_\xi^\infty \Big(\Pi_0^* p_{j-1}(x) + \sum_{i=1}^{j} \Pi_i^* p_{j-i}(x)\Big)\, dH(x) \\ &= \frac{1}{\tau}\Big(\Pi_0^* p_{j-1}(\tau) + \sum_{i=1}^{j} \Pi_i^* p_{j-i}(\tau)\Big). \end{aligned}$$

We conclude that

$$Q_j'(\xi) = \begin{cases} \frac{1}{\tau}\Pi_{j-1}^* & \text{when } \xi \leq \tau, \\ 0 & \text{when } \xi > \tau. \end{cases} \qquad (2)$$

$\boxed{b}$ It is obvious, and follows also from Eqs. (12.2) and (1), that remaining service time has distribution function

$$\tilde{H}(t) = \begin{cases} \frac{t}{\tau} & \text{when } t < \tau, \\ 1 & \text{when } t \geq \tau. \end{cases}$$

According to (12.9),

$$P\{W > t \mid W > 0\} = 1 - \tilde{H}(t) + \sum_{j=2}^{\infty} \frac{j-1}{j} \int_0^t \breve{W}_j(t-\xi)\, Q_j'(\xi)\, d\xi .$$

Substitution of the above expressions for $\tilde{H}(t)$ and $Q_j'(\xi)$ yields

$$P\{W > t \mid W > 0\} = 1 - \frac{t}{\tau} + \frac{1}{\tau}\sum_{j=2}^{\infty} \frac{j-1}{j}\Pi_{j-1}^* \int_0^t \breve{W}_j(t-\xi)\, d\xi \quad (0 \leq t < \tau), \quad (3)$$

$$P\{W > t \mid W > 0\} = \frac{1}{\tau}\sum_{j=2}^{\infty} \frac{j-1}{j}\Pi_{j-1}^* \int_0^\tau \breve{W}_j(t-\xi)\, d\xi \qquad (\tau \leq t < \infty). \quad (4)$$

(Chap. 5, Ex. 29 c)

$\boxed{c}$ Evidently, the waiting time from T_1 and on is a multiple of τ. Hence, $\breve{W}_j(x)$ is constant on each of the intervals $n\tau \leqq x < (n+1)\tau$, $n = 0,1,2,\dots$. In particular, $\breve{W}_j(x) = 1$ for $0 \leqq x < \tau$. Hence,

$$\int_0^t \breve{W}_j(t-\xi)\,d\xi = t \qquad (0 \leqq t < \tau;\ j = 2,3,\dots).$$

so that (3) becomes

$$P\{W>t \mid W>0\} = 1 - \frac{t}{\tau} + \frac{t}{\tau}\sum_{j=2}^{\infty}\frac{j-1}{j}\,\pi_{j-1}^{*} \qquad (0 \leqq t < \tau) \qquad (5)$$

For $t \geqq \tau$ we have, for each $j \geqq 2$,

$$\int_0^{\tau} \breve{W}_j(t-\xi)\,d\xi = \int_{[\frac{t}{\tau}]\tau}^{t} \breve{W}_j(x)\,dx + \int_{t-\tau}^{[\frac{t}{\tau}]\tau} \breve{W}_j(x)\,dx .$$

By constancy of $\breve{W}_j(x)$ on the intervals $n\tau \leqq x < (n+1)\tau$,

$$\int_0^{\tau} \breve{W}_j(t-\xi)\,d\xi = \left(t - [\tfrac{t}{\tau}]\tau\right)\breve{W}_j(t) + \left([\tfrac{t}{\tau}]\tau - (t-\tau)\right)\breve{W}_j(t-\tau)$$
$$(t \geqq \tau;\ j = 2,3,\dots).$$

Hence, (4) becomes

$$P\{W>t \mid W>0\} = \frac{1}{\tau}\sum_{j=2}^{\infty}\frac{j-1}{j}\,\pi_{j-1}^{*}\left\{\left(t-[\tfrac{t}{\tau}]\tau\right)\breve{W}_j(t) + \left([\tfrac{t}{\tau}]\tau-(t-\tau)\right)\breve{W}_j(t-\tau)\right\}$$
$$(\tau \leqq t < \infty). \qquad (6)$$

$\boxed{d}$ Since the service time S_1 is a constant τ, conditional and unconditional distributions of S_1 are identical. By definition of $H(\xi \mid k,x)$ therefore

$$H(\xi \mid k,x) = H(\xi). \qquad (7)$$

$\boxed{e}$ By (1) and (7), $d_\xi H(\xi \mid k,x) = 1$ for $\xi = \tau$, and $d_\xi H(\xi \mid k,x) = 0$ for $\xi \neq \tau$. Consequently, as $k>1$ implies $x \geqq \tau$, Eq. (12.17) reduces to

$$\breve{W}_{j,k}(x) = \sum_{i=0}^{\infty}\frac{j+i-2}{j+i-1}\,p_i(\tau)\,\breve{W}_{j+i-1,k-1}(x-\tau) \qquad \left(\begin{smallmatrix} j = 2,3,\dots;\\ k = 1,2,\dots\end{smallmatrix}\right). \qquad (8)$$

(Chap. 5, Ex. 29 f)

$\boxed{f}$ For constant service time $= \tau$, for all x,

$$\{\breve{X}(x) = k\} \Leftrightarrow \{k\tau \leq x < (k+1)\tau\} \Leftrightarrow k = [\tfrac{x}{\tau}].$$

Hence,

$$P\{\breve{X}(x) = k\} = \begin{cases} 1 & \text{when } k = [\frac{x}{\tau}], \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

$\boxed{g}$ According to (12.13),

$$\breve{W}_j(x) = \sum_{k=0}^{\infty} \breve{W}_{j,k}(x)\, P\{\breve{X}(x) = k\} \qquad (j = 2,3,\ldots).$$

Using (9) we find

$$\breve{W}_j(x) = \breve{W}_{j,[\frac{x}{\tau}]}(x) \qquad (j = 2,3,\ldots). \tag{10}$$

$\boxed{h}$ By setting $k = [\frac{x}{\tau}]$ and $k-1 = [\frac{x-\tau}{\tau}]$ in Eq. (8) is obtained

$$\breve{W}_{j,[\frac{x}{\tau}]}(x) = \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1}\, p_i(\tau)\, \breve{W}_{j+i-1,[\frac{x-\tau}{\tau}]}(x-\tau) \qquad \left(\begin{smallmatrix} j=2,3,\ldots \\ x \geq \tau \end{smallmatrix}\right).$$

The application of (10) leads to

$$\breve{W}_j(x) = \sum_{i=0}^{\infty} \frac{j+i-2}{j+i-1}\, p_i(\tau)\, \breve{W}_{j+i-1}(x-\tau) \qquad (j = 2,3,\ldots;\ x \geq \tau). \tag{11}$$

$\boxed{i}$ It is evident that at the start of service at T_1, any waiting customer will have to wait at least τ time units. That is,

$$\breve{W}_j(x) = 1 \qquad (j = 2,3,\ldots;\ x < \tau). \tag{12}$$

Equations (11) and (12) give $\breve{W}_j(x)$ for $j = 2,3,\ldots$ and all x.

$\boxed{j}$ For $x \geq \tau$ we have $H(x) = 1$. Furthermore, $dH(\xi) = 1$ for $\xi = \tau$, and $dH(\xi) = 0$ for $\xi \neq \tau$. Substitution of these values into Eq. (12.10) also results in Eq. (11).

□

Chapter 5, Exercise 30

'The M/G/1 queue with gating.'

We shall not give all the details of the proof since it is precisely as the proof of (13.14) for the cyclic queue except that $H(x)$ replaces $B(x)$ and $\eta(s)$ replaces $\beta(s)$.

The explanation for this analogy is simple enough. In both cases we seek the mean, $\bar{n}$, of the state distribution of an imbedded Markov chain. Let j be the state variable. If $j \geq 1$ let $\hat{j} = j$, and if $j = 0$ let $\hat{j} = 1$. When $j \geq 1$, service begins (is resumed) right away, but when $j = 0$ service will not begin until a customer arrives. Denote by t the time from start of service until next epoch of the imbedded Markov chain. In the case of the cyclic queue, t is a $\hat{j}$-busy period; in the case of the queue with gating, t is the sum of $\hat{j}$ service times. In either case the state at next epoch will be the number of customers arriving during the mentioned interval of length t, either at the other queue (cyclic queue) or at the same queue (queue with gating). For the M/G/1 queue with gating,

$$P(k) = \sum_{j=1}^{\infty} P(j) \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} dH^{*j}(t) + P(0) \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} dH(t) \quad (k = 0, 1, \ldots), \quad (13.1a)$$

$$g(x) = \sum_{k=0}^{\infty} P(k) x^k, \quad (13.2a)$$

$$g(\eta(\lambda - \lambda x)) - g(x) = P(0)[1 - \eta(\lambda - \lambda x)], \quad (13.5a)$$

$$z_{\nu+1}(x) = z_{\nu+1} = \eta(\lambda - \lambda z_\nu) \quad (\nu = 0, 1, \ldots; z_0 = x), \quad (13.6a)$$

$$P(0) = \{1 + \sum_{j=1}^{\infty} [1 - z_j(0)]\}^{-1}, \quad (13.11a)$$

$$-\lambda \eta'(0)\bar{n} - \bar{n} = \lambda P(0) \eta'(0). \quad (13.13a)$$

By (13.11a) and the definition $x_j = z_j(0)$, $j = 1, 2, \ldots$,

$$P(0) = \{1 + \sum_{j=1}^{\infty} (1 - x_j)\}^{-1}. \quad (*)$$

The mean number $\bar{n} = g(1)$ of customers in the system when the gate opens is found from (13.13a), (*), $-\eta'(0) = \tau$ and $\lambda\tau = \rho$:

$$\bar{n} = \frac{\rho}{1-\rho} \{1 + \sum_{j=1}^{\infty} (1 - x_j)\}^{-1}. \quad \square$$

Chapter 5, Exercise 31

'Show that Equation (44) of Cooper [1969] is incorrect.'

As hinted, the error in Eq. (44) is introduced in Eq. (42). It is true as stated immediately after Eq. (42) that $\hat{P}_i(n) = P_{i-1}(n)/(1-P_{i-1}(0))$ is the probability that n $(n=1,2,\dots)$ customers wait in queue i when the gate closes, given $n \geq 1$. However, it is not true, as implied by Eq. (42), that $\hat{P}_i(n)$ also is the probability that an arbitrary customer in queue i, who did not arrive when the system was completely empty, will be a member of a group of n customers at the time the gate closes. The latter probability is proportional to both n and $\hat{P}_i(n)$. Hence, in Eq. (42) one should replace $P_{i-1}(n)/[1-P_{i-1}(0)]$ by

$$\frac{n\,\hat{P}_i(n)}{\sum_{n=1}^{\infty} n\,\hat{P}_i(n)} = \frac{n\,P_{i-1}(n)}{\sum_{n=1}^{\infty} n\,P_{i-1}(n)}.$$

It follows that the same substitution should take place in Eqs (43) and (44). Equation (44) changes into

$$\omega_i(s) = (1-\rho) + \rho\sum_{n=1}^{\infty} \frac{n\,P_{i-1}(n)}{\sum_{n=1}^{\infty} n\,P_{i-1}(n)}\,\frac{1}{n\,\lambda_i^{n-1}}\,\frac{[\lambda_i\eta_i(s)]^n-(\lambda_i-s)^n}{s-\lambda_i+\lambda_i\eta_i(s)},$$

which may be rewritten as

$$\omega_i(s) = (1-\rho) + \frac{\rho\lambda_i}{(\sum_{n=1}^{\infty} n\,P_{i-1}(n))(s-\lambda_i+\lambda_i\eta_i(s))}\sum_{n=1}^{\infty} P_{i-1}(n)\left[(\eta_i(s))^n-(1-\tfrac{s}{\lambda_i})^n\right].$$

Now,

$$\overline{m}_{i-1} = \frac{\partial}{\partial x}\, g_{i-1}(x,1,\dots,1)\Big|_{x=1} = \sum_{n=1}^{\infty} n\,P_{i-1}(n),$$

$$g_{i-1}(\eta_i(s),1,\dots,1) = P_{i-1}(0) + \sum_{n=1}^{\infty} P_{i-1}(n)\,(\eta_i(s))^n,$$

$$g_{i-1}(1-\tfrac{s}{\lambda_i},1,\dots,1) = P_{i-1}(0) + \sum_{n=1}^{\infty} P_{i-1}(n)\,(1-\tfrac{s}{\lambda_i})^n.$$

We conclude that, for $i=0,1,\dots,N-1$,

$$\omega_i(s) = (1-\rho) + \frac{\rho\lambda_i}{\overline{m}_{i-1}[s-\lambda_i+\lambda_i\eta_i(s)]}\left[g_{i-1}(\eta_i(s),1,\dots,1)-g_{i-1}(1-\tfrac{s}{\lambda_i},1,\dots,1)\right].$$

$\square$

Chapter 5, Exercise 32

'Verify that, for Poisson input, Eq.(14.17) reduces to $P\{W>0\} = C(s,a)$'

With Poisson input at rate λ the Laplace-Stieltjes transform of the interarrival time distribution function is

$$\gamma(z) = \frac{\lambda}{\lambda + z}.$$

Inserting $\lambda/s\mu$ for ω in the right-hand side of (14.12) we get $\gamma((1-\lambda/s\mu)s\mu) = \gamma(s\mu - \lambda) = \lambda/s\mu$, which proves that for the M/M/s queue

$$\omega = \frac{\lambda}{s\mu} = \frac{a}{s}. \qquad (1)$$

Also, (14.15) becomes

$$\gamma_j = \gamma(j\mu) = \frac{\lambda}{\lambda + j\mu} \qquad (j = 0, 1, \ldots, s), \qquad (2)$$

and (14.16) becomes

$$C_j = \prod_{i=1}^{j} \frac{\gamma_i}{1-\gamma_i} = \frac{a^j}{j!} \qquad (j = 1, 2, \ldots, s). \qquad (3)$$

By (14.14), (1), (2) and (3), Eq. (14.17) becomes

$$\begin{aligned} P\{W>0\} &= \frac{A}{1-\omega} \\ &= \Big[(1-\omega)\Big\{\frac{1}{1-\omega} + \sum_{j=1}^{s} \frac{1}{C_j(1-\gamma_j)}\binom{s}{j}\frac{s(1-\gamma_j)-j}{s(1-\omega)-j}\Big\}\Big]^{-1} \\ &= \Big[1 + \Big(1-\frac{a}{s}\Big)\sum_{j=1}^{s} \frac{j!}{a^j}\frac{(\lambda+j\mu)}{j\mu}\,\frac{s!}{j!(s-j)!}\,\frac{\frac{sj\mu}{\lambda+j\mu} - j}{s-a-j}\Big]^{-1} \\ &= \Big[1 + \Big(1-\frac{a}{s}\Big)\sum_{j=1}^{s} \frac{s!}{(s-j)!\,a^j}\Big]^{-1}. \end{aligned}$$

Further rewriting gives

$$P\{W>0\} = \frac{\dfrac{a^s}{s!(1-a/s)}}{\displaystyle\sum_{j=1}^{s} \frac{a^{s-j}}{(s-j)!} + \frac{a^s}{s!(1-a/s)}} = \frac{\dfrac{a^s}{s!(1-a/s)}}{\displaystyle\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!(1-a/s)}}.$$

The rightmost term is precisely the formula for $C(s,a)$ [Eq.(4.8) of Chap. 3]. Thus,

$$P\{W>0\} = C(s,a). \qquad \square$$

Chapter 5, Exercise 33

'Prove that in a GI/M/s queue ...' - cf. Ex. 29 of Chap. 3

P = equilibrium probability that a blocked customer will still be waiting in the queue when the next customer arrives

r_j = conditional probability that a blocked customer will still be waiting in the queue when the next customer arrives, given arrival state $s+j$ $(j=0,1,\dots)$.

Clearly, with service in order of arrival,

$$r_j = \int_0^\infty \sum_{i=0}^{j} \frac{(s\mu x)^i}{i!} e^{-s\mu x}\, dG(x).$$

Hence,

$$\begin{aligned}
P &= \sum_{j=0}^{\infty} r_j P\{Q=j \mid W>0\} \\
&= \sum_{j=0}^{\infty} \left(\int_0^\infty \sum_{i=0}^{j} \frac{(s\mu x)^i}{i!} e^{-s\mu x}\, dG(x) \right) (1-\omega)\omega^j \qquad \text{[by (14.19)]} \\
&= \int_0^\infty e^{-s\mu x} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{(s\mu x)^i}{i!} (1-\omega)\omega^j\, dG(x) \\
&= \int_0^\infty e^{-s\mu x} \sum_{i=0}^{\infty} \frac{(s\mu x)^i}{i!} \sum_{j=i}^{\infty} (1-\omega)\omega^j\, dG(x) \\
&= \int_0^\infty e^{-s\mu x} \sum_{i=0}^{\infty} \frac{(s\mu x\omega)^i}{i!}\, dG(x) \\
&= \int_0^\infty e^{-(1-\omega)s\mu x}\, dG(x) \\
&= \gamma((1-\omega)s\mu).
\end{aligned}$$

But, by (14.11), $\omega = \gamma((1-\omega)s\mu)$. Thus,

$$P = \omega.$$

This generalizes the result of Exercise 29 of Chapter 3. $\square$

Chapter 5, Exercise 34

'Verify Equations (14.7) and (14.8)'

Equation (14.7): $i \leq s-1,\ i+1-j \geq 0$

Given interarrival time x, j has the binomial distribution

$$p_{ij}(x) = \binom{i+1}{j}[e^{-\mu x}]^{j}[1-e^{-\mu x}]^{i+1-j} \quad (0 \leq j \leq i+1),$$

since $p_{ij}(x)$ is the probability of j successes (noncompletions) in i+1 trials, each with probability of success equal to $e^{-\mu x}$. As $P_{ij} = \int_0^\infty p_{ij}(x)\,dG(x)$,

$$P_{ij} = \int_0^\infty \binom{i+1}{j} e^{-j\mu x}(1-e^{-\mu x})^{i+1-j}\,dG(x) \quad \binom{i \leq s-1,}{i+1-j \geq 0}. \qquad (14.7)$$

Equation (14.8): $i \geq s,\ j < s,\ i+1-j \geq 0$

Given interarrival time x, the next arrival state will be j if and only if (i) at some time Y, $0 < Y < x$ (set $T_1 = 0$), a service completion will leave exactly s customers in the system, and (ii) s-j service completions occur in the time interval (Y, x).

In a queue with departure rate $s\mu$ (in effect as long as all servers are busy) the time Y until the (i+1-s)th service completion, which will result in state s, has an Erlangian distribution with the density function, by (5.54) of Chapter 2,

$$\frac{dP\{Y \leq y\}}{dy} = f(y) = \frac{(s\mu y)^{i-s}}{(i-s)!} e^{-s\mu y} s\mu.$$

For a given $Y = y < x$, the probability of s-j service completions during the remaining interarrival interval, of length x-y, equals

$$g_j(x-y) = \binom{s}{j}[e^{-\mu(x-y)}]^{j}[1-e^{-\mu(x-y)}]^{s-j}.$$

Hence, since $p_{ij}(x) = \int_0^x g_j(x-y) f(y)\,dy$ and $P_{ij} = \int_0^\infty p_{ij}(x)\,dG(x)$,

$$P_{ij} = \int_0^\infty \int_0^x \binom{s}{j} e^{-j\mu(x-y)}(1-e^{-\mu(x-y)})^{s-j} \frac{(s\mu y)^{i-s}}{(i-s)!} e^{-s\mu y} s\mu\,dy\,dG(x). \qquad (14.8)$$

$(i \geq s,\ j < s,\ i+1-j \geq 0)$ □

Chapter 5, Exercise 35

'<u>Derivation of (14.14) and the probabilities $\pi_0, \pi_1, \ldots, \pi_{s-2}$</u>'

For the GI/M/s queue it has been shown that $\pi_j = A\omega^{j-s}$ for $j \geq s-1$, see (14.10), and now we must prove that A is given by (14.14). At the same time we derive a formula for π_j for $j = 0, 1, \ldots, s-2$.

$\boxed{a}$ Let

$$U(z) = \sum_{j=0}^{s-1} \pi_j z^j. \qquad (1)$$

Substitution of $\pi_j = \sum_{i=0}^{\infty} p_{ij}\pi_i$, by (14.3), and change of the order of summation give

$$U(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{s-1} p_{ij}\pi_i z^j,$$

or,

$$U(z) = \sum_{i=0}^{s-1} \sum_{j=0}^{s-1} p_{ij}\pi_i z^j + \sum_{i=s}^{\infty} \sum_{j=0}^{s-1} p_{ij}\pi_i z^j. \qquad (*)$$

<u>Calculation of $S = \sum_{i=0}^{s-1} \sum_{j=0}^{s-1} p_{ij}\pi_i z^j$</u>

Since $p_{ij} = 0$ for $j > i+1$,

$$S = \sum_{i=0}^{s-1} \sum_{j=0}^{i+1} p_{ij}\pi_i z^j - p_{s-1,s}\pi_{s-1} z^s.$$

The p_{ij}'s for $i \leq s-1$ and $j \leq i+1$ are given by (14.7). Substitution of (14.7) and interchange of summations and integration yield

$$S_1 = \sum_{i=0}^{s-1} \sum_{j=0}^{i+1} p_{ij}\pi_i z^j = \int_0^{\infty} \sum_{i=0}^{s-1} \pi_i \sum_{j=0}^{i+1} \binom{i+1}{j} [e^{-\mu x}]^j [1-e^{-\mu x}]^{i+1-j} z^j \, dG(x).$$

The inner sum is the probability generating function of a binomial variable and equals $(q+pz)^{i+1}$, where $p = e^{-\mu x}$, $q = 1-e^{-\mu x}$. Thus,

$$S_1 = \int_0^{\infty} \sum_{i=0}^{s-1} \pi_i (1 - e^{-\mu x} + z e^{-\mu x})^{i+1} dG(x).$$

By definition of $U(z)$, then

$$S_1 = \int_0^{\infty} (1 - e^{-\mu x} + z e^{-\mu x}) U(1 - e^{-\mu x} + z e^{-\mu x}) \, dG(x).$$

(Chap. 5, Ex. 35a)

By (14.6) or (14.7), $p_{s-1,s} = \int_0^\infty e^{-s\mu x} dG(x)$, and by (14.10), $\Pi_{s-1} = A\omega^{-1}$. Hence,

$$S_2 = p_{s-1,s}\Pi_{s-1} z^s = A\omega^{-1} z^s \int_0^\infty e^{-s\mu x} dG(x).$$

As $S = S_1 - S_2$,

$$S = \int_0^\infty (1-e^{-\mu x}+ze^{-\mu x})\, U(1-e^{-\mu x}+ze^{-\mu x})\, dG(x) - A\omega^{-1}z^s \int_0^\infty e^{-s\mu x} dG(x) \quad (**)$$

Calculation of $T = \sum_{i=s}^\infty \sum_{j=0}^{s-1} p_{ij}\Pi_i z^j$

The p_{ij}'s for $i \geq s$ and $j < s$ are given by (14.8). Substitution of (14.8) and interchange of summations and integrations yield

$$T = \int_{x=0}^\infty \int_{y=0}^x \left\{ \sum_{i=s}^\infty \sum_{j=0}^{s-1} \binom{s}{j} [e^{-\mu(x-y)}]^j [1-e^{-\mu(x-y)}]^{s-j} \frac{(s\mu y)^{i-s}}{(i-s)!} e^{-s\mu y} s\mu \Pi_i z^j \right\} dy\, dG(x).$$

Substitution of $\Pi_i = A\omega^{i-s}$, by (14.10), rewriting and simplification give

$$T = A\int_{x=0}^\infty \int_{y=0}^x \sum_{i=s}^\infty \frac{(s\mu y\omega)^{i-s}}{(i-s)!} e^{-s\mu y} \left\{ \sum_{j=0}^{s} \binom{s}{j} [e^{-\mu(x-y)}]^j [1-e^{-\mu(x-y)}]^{s-j} z^j - e^{-s\mu(x-y)} z^s \right\} \cdot s\mu\, dy\, dG(x)$$

$$= A\int_{x=0}^\infty \int_{y=0}^x e^{s\mu y\omega} e^{-s\mu y} (1-e^{-\mu(x-y)}+ze^{-\mu(x-y)})^s s\mu\, dy\, dG(x)$$

$$- A z^s \int_{x=0}^\infty e^{-s\mu x} \int_{y=0}^x s\mu\, e^{s\mu y\omega} dy\, dG(x)$$

$$= A\int_{x=0}^\infty \int_{y=0}^x e^{s\mu y\omega} (e^{-\mu y} - e^{-\mu x} + ze^{-\mu x})^s s\mu\, dy\, dG(x)$$

$$- A z^s \int_0^\infty e^{-s\mu x} \omega^{-1} (e^{s\mu x\omega} - 1)\, dG(x)$$

The last integral on the right-hand side reduces to

$$\int_0^\infty e^{-s\mu x}\omega^{-1}(e^{s\mu x\omega}-1)\, dG(x) = \omega^{-1}\int_0^\infty e^{-(1-\omega)s\mu x} dG(x) - \omega^{-1}\int_0^\infty e^{-s\mu x} dG(x)$$

$$= 1 - \omega^{-1}\int_0^\infty e^{-s\mu x} dG(x), \qquad \text{[by (14.11)]}$$

so that

$$T = A\int_{x=0}^\infty \int_{y=0}^x e^{s\mu y\omega}(e^{-\mu y} - e^{-\mu x} + ze^{-\mu x})^s s\mu\, dy\, dG(x) - Az^s + A\omega^{-1}z^s \int_0^\infty e^{-s\mu x} dG(x). \quad (***)$$

(Chap. 5, Ex. 35a (cont'd))

By (*), (**), (***), and the definitions of S and T,

$$U(z) = \int_0^\infty (1-e^{-\mu x}+ze^{-\mu x})\,U(1-e^{-\mu x}+ze^{-\mu x})\,dG(x)$$
$$+A\int_{x=0}^\infty\Big[\int_{y=0}^x e^{s\mu\omega y}(e^{-\mu y}-e^{-\mu x}+ze^{-\mu x})^s s\mu\,dy\Big]dG(x)$$
$$-Az^s. \qquad (2)$$

$\boxed{b}$ By (1) and (14.10),

$$U(1) = \sum_{j=0}^{s-1}\pi_j = 1-\sum_{j=s}^\infty \pi_j = 1-\sum_{j=s}^\infty A\,\omega^{j-s}.$$

Hence,

$$U(1) = 1-\frac{A}{1-\omega}. \qquad (3)$$

$\boxed{c}$ For $j=0,1,\dots,s-1$ let

$$U^{(j)}(z) = \frac{d^jU(z)}{dz^j},$$

with $U^{(0)}(z) = U(z)$, and define

$$U_j = \frac{U^{(j)}(1)}{j!} = \frac{1}{j!}\Big(\frac{d^jU(z)}{dz^j}\Big)_{z=1} \qquad (j=0,1,\dots,s-1). \qquad (4)$$

By definition, $U_0 = \frac{1}{0!}U(1) = U(1)$. By (3), therefore,

$$U_0 = 1-\frac{A}{1-\omega}. \qquad (5)$$

Repeated differentiation of (2) gives

$$U^{(j)}(z) = \int_0^\infty (1-e^{-\mu x}+ze^{-\mu x})\,U^{(j)}(1-e^{-\mu x}+ze^{-\mu x})\,e^{-j\mu x}dG(x)$$
$$+j\int_0^\infty U^{(j-1)}(1-e^{-\mu x}+ze^{-\mu x})\,e^{-j\mu x}dG(x)$$
$$+As\mu\int_{x=0}^\infty e^{-j\mu x}\int_{y=0}^x e^{s\mu\omega y}\frac{s!}{(s-j)!}(e^{-\mu y}-e^{-\mu x}+ze^{-\mu x})^{s-j}dy\,dG(x)$$
$$-A\frac{s!}{(s-j)!}z^{s-j} \qquad (j=1,2,\dots,s-1).$$

(Chap. 5, Ex. 35 c)

Hence,

$$U_j = \frac{U^{(j)}(1)}{j!} = U_j\,\gamma(j\mu) + U_{j-1}\,\gamma(j\mu)$$
$$+ A\binom{s}{j} s\mu \int_{x=0}^{\infty} e^{-j\mu x} \int_{y=0}^{x} e^{(s\mu\omega-(s-j)\mu)y}\,dy\,dG(x)$$
$$- A\binom{s}{j} \qquad (j = 1, 2, \ldots, s-1).$$

Substituting

$$\int_0^x e^{(s\mu\omega-(s-j)\mu)y}\,dy = \frac{e^{(s\mu\omega-(s-j)\mu)x}-1}{s\mu\omega-(s-j)\mu}$$

and reducing, we obtain

$$U_j = U_j\,\gamma(j\mu) + U_{j-1}\,\gamma(j\mu)$$
$$- A\binom{s}{j}\frac{s}{s(1-\omega)-j}\left(\int_0^\infty e^{-(1-\omega)s\mu x}\,dG(x) - \int_0^\infty e^{-j\mu x}\,dG(x)\right)$$
$$- A\binom{s}{j} \qquad (j = 1, 2, \ldots, s-1).$$

The two integrals are equal to, respectively, ω (by (14.11)) and $\gamma(j\mu)$. Substitution of these values, simplification, and replacement of $\gamma(j\mu)$ by γ_j, give

$$U_j = U_j\,\gamma_j + U_{j-1}\,\gamma_j - A\binom{s}{j}\frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \qquad (j = 1, 2, \ldots, s-1),$$

from which is obtained the difference equation

$$U_j = \frac{\gamma_j}{1-\gamma_j}U_{j-1} - \frac{A}{1-\gamma_j}\binom{s}{j}\frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (j=1,2,\ldots,s-1). \qquad (6)$$

d Next we define $C_0 = 1$ and

$$C_j = \prod_{i=1}^{j}\frac{\gamma_i}{1-\gamma_i} \qquad (j = 1, 2, \ldots, s),$$

and divide by C_j on both sides of (6). The result is

$$\frac{U_j}{C_j} = \frac{U_{j-1}}{C_{j-1}} - \frac{A}{C_j(1-\gamma_j)}\binom{s}{j}\frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (j = 1, 2, \ldots, s-1). \qquad (7)$$

(Chap 5, Ex 35d)

Assuming $0 \le i \le s-2$ we add equations (7) for $j = i+1, \ldots, s-1$, whereby we derive

$$\frac{U_i}{C_i} = \frac{U_{s-1}}{C_{s-1}} + A \sum_{j=i+1}^{s-1} \frac{1}{C_j(1-\gamma_j)} \binom{s}{j} \frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (i = 0, 1, \ldots, s-2),$$

$$\frac{U_i}{C_i} = \left(\frac{U_{s-1}}{C_{s-1}} - \frac{A\omega^{-1}}{C_s(1-\gamma_s)/\gamma_s}\right) + A \sum_{j=i+1}^{s} \frac{1}{C_j(1-\gamma_j)} \binom{s}{j} \frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (i = 0, 1, \ldots, s-2).$$

By (1) and (4), $U_{s-1} = \Pi_{s-1}$, and, by (14.10), $\Pi_{s-1} = A\omega^{-1}$. By definition, $C_s = C_{s-1}\gamma_s/(1-\gamma_s)$. It follows that the term in parentheses vanishes. Furthermore, as is easily verified, the above equation also holds for $i = s-1$. Our conclusion is that

$$\frac{U_i}{C_i} = A \sum_{j=i+1}^{s} \frac{1}{C_j(1-\gamma_j)} \binom{s}{j} \frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \quad (i = 0, 1, \ldots, s-1). \qquad (8)$$

Now set $i = 0$, make the substitutions $U_0 = 1 - \frac{A}{1-\omega}$ and $C_0 = 1$, and solve Eq. (8) for A:

$$A = \left\{ \frac{1}{1-\omega} + \sum_{j=1}^{s} \frac{1}{C_j(1-\gamma_j)} \binom{s}{j} \frac{s(1-\gamma_j)-j}{s(1-\omega)-j} \right\}^{-1}. \qquad (14.14)$$

$\boxed{e}$ By definition, $U(z)$ is a polynomial in z of degree $s-1$. Hence, $U(z)$ will be represented exactly by a Taylor series expansion of degree $s-1$ at an arbitrary point z_0. For $z_0 = 1$ the representation is $U(z) = \sum_{j=0}^{s-1} [U^{(j)}(1)/j!](z-1)^j$. That is,

$$U(z) = \sum_{j=0}^{s-1} U_j (z-1)^j. \qquad (9)$$

By (1), $U^{(j)}(z) = \sum_{k=j}^{s-1} \Pi_k \frac{k!}{(k-j)!} z^{k-j}$. Hence, $U^{(j)}(0) = \Pi_j\, j!$, or,

$$\Pi_j = \frac{U^{(j)}(0)}{j!} = \frac{1}{j!}\left(\frac{d^jU(z)}{dz^j}\right)_{z=0} \quad (j = 0, 1, \ldots, s-1). \qquad (10)$$

Differentiation of (9) $[U(z) = \sum_{i=0}^{s-1} U_i (z-1)^i]$ j times leads to

$$\frac{d^jU(z)}{dz^j} = \sum_{i=j}^{s-1} U_i \frac{i!}{(i-j)!} (z-1)^{i-j} \quad (j = 0, 1, \ldots, s-1),$$

whose substitution into (10) yields

$$\Pi_j = \sum_{i=j}^{s-1} (-1)^{i-j} \binom{i}{j} U_i \quad (j = 0, 1, \ldots, s-2, s-1). \qquad (11) \quad \square$$

Chapter 5, Exercise 36

'Show that for Poisson input this algorithm yields Equation (6) of Exercise 32 of Chapter 3.'

In the case of the M/M/s queue with service in random order, $\omega = \rho$ $(=\lambda/s\mu)$, so that (15.9) becomes

$$P\{W>t \mid W>0\} = 1 + (1-\rho)\sum_{\nu=1}^{\infty}\frac{t^{\nu}}{\nu!}\sum_{j=0}^{\infty}\rho^{j}W_j^{(\nu)}, \tag{1}$$

where, according to (15.14),

$$W_j^{(\nu)} = a_{j+1}^{(\nu)} + \sum_{i=1}^{j+1}\frac{i}{j+1}\sum_{k=0}^{\nu-1} b_{j+1-i}^{(k)}\, W_i^{(\nu-1-k)} \qquad \left(\begin{matrix} j=0,1,\ldots; \\ \nu=1,2,\ldots \end{matrix}\right). \tag{2}$$

Eq. (2) may be solved recursively for $\nu=1$, $\nu=2,\ldots$ utilizing $W_j^{(0)} = W_j(0) = 1$ and the definitions of $a_{j+1}(t)$, by (15.12), and $b_{j+1-i}(t)$, by (15.13). In the present case where $G(t) = 1-e^{-\lambda t}$,

$$a_{j+1}(t) = e^{-(\lambda+s\mu)t}\sum_{i=1}^{j+1}\frac{i}{j+1}\frac{(s\mu t)^{j+1-i}}{(j+1-i)!} \qquad (j=0,1,\ldots), \tag{3}$$

$$b_n(t) = \lambda e^{-(\lambda+s\mu)t}\frac{(s\mu t)^n}{n!} \qquad (n=0,1,\ldots), \tag{4}$$

from which $a_{j+1}^{(\nu)}$ and $b_{j+1-i}^{(k)}$ may be derived.

We shall determine all terms of the expansion (1) to the order of t^2. Thus we must find $W_j^{(1)}$ and $W_j^{(2)}$ for $j=0,1,\ldots$. By (2), this requires calculation of $a_{j+1}^{(\nu)}$ for $\nu=1,2$, and of $b_{j+1-i}^{(k)}$ for $k=0,1$.

Calculation of $a_{j+1}^{(1)}$ and $a_{j+1}^{(2)}$

Differentiating (3) once, we obtain

$$\begin{aligned}
a_1^{(1)}(t) &= -(\lambda+s\mu)e^{-(\lambda+s\mu)t}, \\
a_2^{(1)}(t) &= -(\lambda+s\mu)e^{-(\lambda+s\mu)t}\left[1-\frac{1}{1+\rho}\frac{1}{2}+\frac{1}{2}(s\mu t)\right], \\
a_{j+1}^{(1)}(t) &= -(\lambda+s\mu)e^{-(\lambda+s\mu)t}\left[1-\frac{1}{1+\rho}\frac{j}{j+1}+\sum_{i=1}^{j}\frac{i}{j+1}\frac{(s\mu t)^{j+1-i}}{(j+1-i)!}\right. \\
&\qquad \left. -\frac{1}{1+\rho}\sum_{i=1}^{j-1}\frac{i}{j+1}\frac{(s\mu t)^{j-i}}{(j-i)!}\right] \quad (j=2,3,\ldots).
\end{aligned}$$

Setting $t=0$, we find $a_{j+1}^{(1)} = -(\lambda+s\mu)\left[1-\frac{1}{1+\rho}\frac{j}{j+1}\right]$ for $j=0,1,2$, or,

$$a_{j+1}^{(1)} = -s\mu\left(\rho+\frac{1}{j+1}\right) \qquad (j=0,1,\ldots). \tag{5}$$

(Chap. 5, Ex. 36)

Another differentiation results in

$$a_1^{(2)}(t) = (\lambda+s\mu)^2 e^{-(\lambda+s\mu)t},$$
$$a_2^{(2)}(t) = (\lambda+s\mu)^2 e^{-(\lambda+s\mu)t}\left[1-\frac{1}{1+\rho}+\frac{1}{2}(s\mu t)\right],$$
$$a_{j+1}^{(2)}(t) = (\lambda+s\mu)^2 e^{-(\lambda+s\mu)t}\left[1-\frac{2}{1+\rho}\frac{j}{j+1}+\frac{1}{(1+\rho)^2}\frac{j-1}{j+1}+tR_{j-1}(t)\right] \quad (j=2,3,\ldots),$$

where $R_{j-1}(t)$ is a polynomium of degree $j-1$. It follows that

$$a_1^{(2)} = (\lambda+s\mu)^2,$$
$$a_2^{(2)} = (\lambda+s\mu)^2\left[1-\frac{1}{1+\rho}\right],$$
$$a_{j+1}^{(2)} = (\lambda+s\mu)^2\left[1-\frac{2}{1+\rho}\frac{j}{j+1}+\frac{1}{(1+\rho)^2}\frac{j-1}{j+1}\right] \quad (j=2,3,\ldots).$$

A rewriting yields

$$a_{j+1}^{(2)} = \begin{cases} (s\mu)^2(\rho^2+2\rho+1) & (j=0), \\ (s\mu)^2(\rho^2+\frac{2\rho}{j+1}) & (j=1,2,\ldots). \end{cases} \tag{6}$$

Calculation of $b_{j+1-i}^{(0)}$ and $b_{j+1-i}^{(1)}$ —

By (4), clearly $b_0^{(0)} = \lambda$ and $b_n^{(0)} = 0$ for $n \geq 1$. Thus,

$$b_{j+1-i}^{(0)} = \begin{cases} \lambda & (j+1-i=0), \\ 0 & (j+1-i=1,2,\ldots). \end{cases} \tag{7}$$

Differentiation of (4) results in

$$b_0^{(1)}(t) = -\lambda(\lambda+s\mu)e^{-(\lambda+s\mu)t},$$
$$b_n^{(1)}(t) = \lambda e^{-(\lambda+s\mu)t}\left[\frac{(s\mu t)^{n-1}}{(n-1)!}(s\mu) - \frac{(s\mu t)^n}{n!}(\lambda+s\mu)\right] \quad (n=1,2,\ldots).$$

It follows that $b_0^{(1)} = -\lambda(\lambda+s\mu)$, $b_1^{(1)} = \lambda s\mu$, $b_n^{(1)} = 0$ for $n \geq 2$. Thus,

$$b_{j+1-i}^{(1)} = \begin{cases} -\lambda(\lambda+s\mu) & (j+1-i=0), \\ \lambda s\mu & (j+1-i=1), \\ 0 & (j+1-i=2,3,\ldots). \end{cases} \tag{8}$$

(Chap. 5, Ex. 36 (cont'd))

Calculation of $W_j^{(1)}$

By (2),

$$W_j^{(1)} = a_{j+1}^{(1)} + \sum_{i=1}^{j+1} \frac{i}{j+1} b_{j+1-i}^{(0)} W_i^{(0)} \quad (j = 0,1,\ldots).$$

Substitution of $W_i^{(0)} = 1$, (5), and (7), leads to $W_j^{(1)} = -s\mu[\rho + \frac{1}{j+1}] + \lambda$ for $j = 0,1,\ldots$. Hence,

$$W_j^{(1)} = -\frac{s\mu}{j+1} \quad (j = 0,1,\ldots). \qquad (9)$$

Calculation of $W_j^{(2)}$

By (2),

$$W_j^{(2)} = a_{j+1}^{(2)} + \sum_{i=1}^{j+1} \frac{i}{j+1} b_{j+1-i}^{(0)} W_i^{(1)} + \sum_{i=1}^{j+1} \frac{i}{j+1} b_{j+1-i}^{(1)} W_i^{(0)} \quad (j = 0,1,\ldots),$$

which by substitution of $W_i^{(0)} = 1$, (7) and (9) reduces to

$$W_j^{(2)} = a_{j+1}^{(2)} - \frac{\lambda s\mu}{j+2} + \frac{1}{j+1} \sum_{i=1}^{j+1} i\, b_{j+1-i}^{(1)} \qquad (j = 0,1,\ldots).$$

Finally, application of (6) and (8) results in

$$W_0^{(2)} = (s\mu)^2(\rho^2 + 2\rho + 1) - \frac{\lambda s\mu}{2} - \lambda(\lambda + s\mu),$$
$$W_j^{(2)} = (s\mu)^2(\rho^2 + \frac{2\rho}{j+1}) - \frac{\lambda s\mu}{j+2} + \frac{j}{j+1}\lambda s\mu - \lambda(\lambda + s\mu) \quad (j = 1,2,\ldots),$$

or,

$$W_j^{(2)} = \begin{cases} (s\mu)^2(1 + \frac{\rho}{2}) & (j = 0), \\ (s\mu)^2 \frac{\rho}{(j+1)(j+2)} & (j = 1,2,\ldots). \end{cases} \qquad (10)$$

Calculation of $P\{W > t \mid W > 0\}$

Equations (9) and (10) are precisely those derived previously in Exercise 32 of Chapter 3. As in that exercise, the substitution of (9) and (10) into (1) and subsequent reduction therefore yield Equation (6) of Exercise 32 of Chapter 3 $\square$