Chapter 6, Exercise 1

'Show that if U is uniform on $(0,1)$, then so is $1-U$.'

Assume that

$$P\{U \leq u\} = u \qquad (0 \leq u \leq 1).$$

Hence,

$$P\{U < u\} = u \qquad (0 \leq u \leq 1).$$

Now, $\{U < u\} \Leftrightarrow \{1-U > 1-u\}$, so that

$$P\{1-U > 1-u\} = u \qquad (0 \leq u \leq 1),$$

whereby

$$P\{1-U \leq 1-u\} = 1-u \quad (0 \leq u \leq 1).$$

Substituting $u' = 1-u$ we obtain

$$P\{1-U \leq u'\} = u' \qquad (0 \leq u' \leq 1).$$

Thus, $1-U$ is uniformly distributed on $(0,1)$ if U is.

Chapter 6, Exercise 2

'Let X have the Erlangian distribution function of order n,'

$$F_X(x) = 1 - \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x}.$$

X may be interpreted as the sum of n independent exponential variables with parameter λ, that is,

$$X = X_1 + X_2 + \dots + X_n,$$

where

$$F_{X_i}(x) = 1 - e^{-\lambda x} \quad (i = 1, \dots, n).$$

But, by (2.7), $X_i = -\frac{1}{\lambda} \ln U_i$, where U_i is uniform on $(0,1)$. Hence,

$$X = \sum_{i=1}^{n} \left(-\frac{1}{\lambda} \ln U_i\right) = -\frac{1}{\lambda} \ln (U_1 U_2 \dots U_n). \qquad \square$$

Chapter 6, Exercise 3

'Derive Equation (4.18).'

The variable under consideration is

$$\hat{P}(n) = \frac{\overline{X}(n)}{\overline{Y}(n) + \overline{X}(n)}, \qquad (1)$$

with $\overline{X}(n) = \sum_{i=1}^{n} X_i/n$, $\overline{Y}(n) = \sum_{i=1}^{n} Y_i/n$, where all X_i's and Y_i's are independent variables. The service time X_i has a general distribution, $E(X_i) = \tau$, $V(X_i) = \sigma^2$. The idle time Y_i has an exponential distribution, $E(Y_i) = \lambda^{-1}$, $V(Y_i) = \lambda^{-2}$.

Now define the distribution function

$$F_{\hat{P}(n)}(t) = P\{\hat{P}(n) \leqq t\}. \qquad (2)$$

Substitution of (1) into (2) gives

$$F_{\hat{P}(n)}(t) = P\{\overline{X}(n) - \tfrac{t}{1-t}\overline{Y}(n) \leqq 0\}. \qquad (3)$$

Defining $Z_i(t) = X_i - \frac{t}{1-t} Y_i$ and $\overline{Z}(n;t) = \sum_{i=1}^{n} Z_i(t)/n = \overline{X}(n) - \frac{t}{1-t}\overline{Y}(n)$, (3) may be written

$$F_{\hat{P}(n)}(t) = P\{\overline{Z}(n;t) \leqq 0\}. \qquad (4)$$

We find easily that mean and variance of $\overline{Z}(n;t)$ are

$$E(\overline{Z}(n;t)) = \tau - \tfrac{t}{1-t}\lambda^{-1}, \quad V(\overline{Z}(n;t)) = \tfrac{1}{n}[\sigma^2 + (\tfrac{t}{1-t})^2\lambda^{-2}]. \qquad (5), (6)$$

By the central limit theorem $\overline{Z}(n;t)$ is asymptotically normal distributed:

$$\lim_{n\to\infty} P\{\overline{Z}(n;t) \leqq x\} = \Phi\left(\frac{x - E(\overline{Z}(n;t))}{\sqrt{V(\overline{Z}(n;t))}}\right). \qquad (7)$$

Setting $x = 0$, and substituting (5) and (6) we derive

$$\lim_{n\to\infty} P\{\overline{Z}(n;t) \leqq 0\} = \Phi\left(\frac{\frac{t}{1-t} - a}{\sqrt{\frac{1}{n}[(\frac{t}{1-t})^2 + \lambda^2\sigma^2]}}\right).$$

By (4),

$$\lim_{n\to\infty} F_{\hat{P}(n)}(t) = \Phi\left(\frac{\frac{t}{1-t} - a}{\sqrt{\frac{1}{n}[(\frac{t}{1-t})^2 + \lambda^2\sigma^2]}}\right). \qquad (4.18)$$

$\square$

Chapter 6, Exercise 4

'Consider simulation of the single-server Erlang loss model with constant service times.'

Our two estimates $\hat{\pi}(n)$ and $\hat{P}(n)$ of the loss probability π, based on a simulation of n cycles, have been shown to be asymptotically normal:

$$F_{\hat{\pi}(n)}(t) \simeq \Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}(a+\lambda^2\sigma^2)}}\right), \qquad (4.11)$$

$$F_{\hat{P}(n)}(t) \simeq \Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}[(\frac{t}{1-t})^2+\lambda^2\sigma^2]}}\right). \qquad (4.18)$$

[a] First, observe that $\pi = \frac{\tau}{\lambda^{-1}+\tau} = \frac{a}{1+a}$, whereby $a = \frac{\pi}{1-\pi}$. Also, as $n \to \infty$, both $\hat{\pi}(n)$ and $\hat{P}(n)$ converge in probability to π. It is therefore obvious, and may be proved rigorously, that for any $\varepsilon > 0$ there exists an n_0 such that

$$\Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}[(\frac{t}{1-t})^2+\lambda^2\sigma^2]}}\right) - \Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}(a^2+\lambda^2\sigma^2)}}\right) < \varepsilon$$

for all t, $0 < t < 1$, and $n > n_0$. This means that asymptotically $\hat{P}(n)$'s distribution function may be also expressed

$$F_{\hat{P}(n)}(t) \simeq \Phi\left(\frac{\frac{t}{1-t}-a}{\sqrt{\frac{1}{n}(a^2+\lambda^2\sigma^2)}}\right). \qquad (4.18a)$$

For $a=1$, a comparison of (4.11) and (4.18a) leads to the conclusion that, asymptotically, the two probability distributions are the same, $F_{\hat{\pi}(n)}(t) \simeq F_{\hat{P}(n)}(t)$. Hence, for all values of δ,

$$P\{\pi-\delta < \hat{\pi}(n) < \pi+\delta\} \simeq P\{\pi-\delta < \hat{P}(n) < \pi+\delta\} \quad (a=1).$$

[b] Now assume constant service times, that is $\sigma^2 = 0$, and $n = 100$. Substituting these values and $a = \frac{\pi}{1-\pi}$ into (4.11) and (4.18) we find

$$F_{\hat{\pi}(100)}(t) \simeq \Phi\left(10\left(\frac{t}{1-t}-\frac{\pi}{1-\pi}\right)\sqrt{\frac{1-\pi}{\pi}}\right), \qquad (1)$$

$$F_{\hat{P}(100)}(t) \simeq \Phi\left(10\left(1-\frac{\pi}{1-\pi}\,\frac{1-t}{t}\right)\right). \qquad (2)$$

(Chap. 6, Ex. 4)

Hence we obtain the approximation formulas

$$P\{\pi - 0.05 < \hat{\pi}(100) < \pi + 0.05\} = \Phi(u_1) - \Phi(u_2), \tag{3}$$

$$P\{\pi - 0.05 < \hat{P}(100) < \pi + 0.05\} = \Phi(u_3) - \Phi(u_4), \tag{4}$$

where

$$u_1 = 10\left[\frac{\pi+0.05}{1-(\pi+0.05)} - \frac{\pi}{1-\pi}\right]\sqrt{\frac{1-\pi}{\pi}},$$

$$u_2 = -10\left[\frac{\pi}{1-\pi} - \frac{\pi-0.05}{1-(\pi-0.05)}\right]\sqrt{\frac{1-\pi}{\pi}},$$

$$u_3 = 10\left[1 - \frac{\pi}{1-\pi}\,\frac{1-(\pi+0.05)}{\pi+0.05}\right],$$

$$u_4 = -10\left[\frac{\pi}{1-\pi}\,\frac{1-(\pi-0.05)}{\pi-0.05} - 1\right].$$

Calculations

Table 1. Arguments of $\Phi(\cdot)$

π	u_1	u_2	u_3	u_4
0.05	2.55	− 2.29	5.26	− ∞
0.10	1.96	− 1.75	3.70	−11.11
0.15	1.75	− 1.56	2.94	− 5.88
0.20	1.67	− 1.47	2.50	− 4.17
0.25	1.65	− 1.44	2.22	− 3.33
0.30	1.68	− 1.45	2.04	− 2.86
0.35	1.75	− 1.50	1.92	− 2.56
0.40	1.86	− 1.57	1.85	− 2.38
0.45	2.01	− 1.68	1.82	− 2.27
0.50	2.22	− 1.82	1.82	− 2.22
0.55	2.51	− 2.01	1.85	− 2.22
0.60	2.92	− 2.27	1.92	− 2.27
0.65	3.44	− 2.62	2.04	− 2.38
0.70	4.36	− 3.12	2.22	− 2.56
0.75	5.77	− 3.85	2.50	− 2.86
0.80	8.33	− 5.00	2.94	− 3.33
0.85	14.00	− 7.00	3.70	− 4.17
0.90	33.33	− 11.11	5.26	− 5.88
0.95	∞	− 22.94	10.00	−11.11

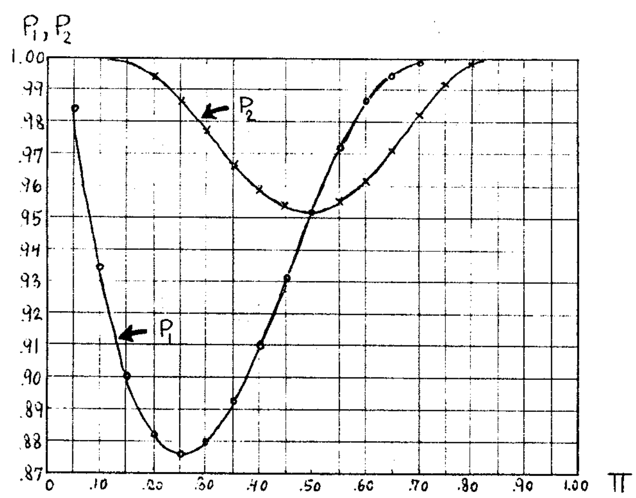
(Chap. 6, Ex. 4 (cont'd))

Table 2. $P_1 = P\{\pi - 0.05 < \hat{\pi}(100) < \pi + 0.05\}$

π	$\Phi(u_1)$	$\Phi(u_2)$	P_1
0.05	.9946	.0110	.984
0.10	.9750	.0401	.935
0.15	.9594	.0594	.900
0.20	.9525	.0708	.882
0.25	.9505	.0749	.876
0.30	.9535	.0735	.880
0.35	.9599	.0668	.893
0.40	.9686	.0582	.910
0.45	.9778	.0465	.931
0.50	.9868	.0344	.952
0.55	.9940	.0222	.972
0.60	.9982	.0116	.987
0.65	.9998	.0044	.995
0.70	1.0000	.0009	.999
0.75	1.0000	.0000	1.000
0.80	1.0000	.0000	1.000
0.85	1.0000	.0000	1.000
0.90	1.0000	.0000	1.000
0.95	1.0000	.0000	1.000

Table 3 $P_2 = P\{\pi - 0.05 < \hat{P}(100) < \pi + 0.05\}$

π	$\Phi(u_3)$	$\Phi(u_4)$	P_2
0.05	1.0000	.0000	1.000
0.10	.9999	.0000	1.000
0.15	.9984	.0000	.998
0.20	.9938	.0000	.994
0.25	.9868	.0004	.986
0.30	.9793	.0021	.977
0.35	.9726	.0052	.967
0.40	.9678	.0087	.959
0.45	.9656	.0116	.954
0.50	.9656	.0132	.952
0.55	.9678	.0132	.955
0.60	.9726	.0116	.961
0.65	.9793	.0087	.971
0.70	.9868	.0052	.982
0.75	.9938	.0021	.992
0.80	.9984	.0004	.998
0.85	.9999	.0000	1.000
0.90	1.0000	.0000	1.000
0.95	1.0000	.0000	1.000



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