Chapter 3, Exercise 1

'A single-server queueing system...'

$\boxed{a}$ By Equation (1.1),

$$P_0 = [1 + \sum_{k=1}^{\infty} \frac{\lambda_0\lambda_1 \cdots \lambda_{k-1}}{\mu_1\mu_2 \cdots \mu_k}]^{-1} = [1 + \sum_{k=1}^{\infty} \frac{(\lambda/\mu)^k}{k!}]^{-1} = e^{-\lambda/\mu},$$

and for $j = 1, 2, \ldots$

$$P_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j} P_0 = \frac{(\lambda/\mu)^j}{j!} e^{-\lambda/\mu}.$$

Since $\mu^{-1} = \tau$,

$$P_j = \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} \qquad (j = 0, 1, \ldots).$$

By Equation (2.6), for $j = 0, 1, \ldots,$

$$\Pi_j = \lambda_j P_j / \sum_{k=0}^{\infty} \lambda_k P_k = \frac{\lambda}{j+1} \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} / \sum_{k=0}^{\infty} \frac{\lambda}{k+1} \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau} = \frac{\frac{(\lambda\tau)^{j+1}}{(j+1)!} e^{-\lambda\tau}}{\sum_{k=0}^{\infty} \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau} - e^{-\lambda\tau}}.$$

Hence,

$$\Pi_j = (1 - e^{-\lambda\tau})^{-1} P_{j+1} \quad (j = 0, 1, \ldots).$$

As $s = 1$, by (1.9) the carried load is

$$a' = \sum_{j=1}^{\infty} P_j = 1 - P_0 = 1 - e^{-\lambda\tau}.$$

Obviously, the mean arrival rate is $\bar{\lambda} = \Sigma_{j=0}^{\infty} \lambda_j P_j$, so

$$\bar{\lambda} = \sum_{j=0}^{\infty} \frac{\lambda}{j+1} \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} = \tau^{-1}(1 - e^{-\lambda\tau}).$$

It follows that the offered load is $a = \bar{\lambda}\tau = 1 - e^{-\lambda\tau}$, and $a = a'$.

$\boxed{b}$ Here the arrival rate is λ, but the effective arrival rate — concerning arrivals effecting a change of state — in state j is $\lambda_j = \lambda[1 - j/(j+1)] = \lambda/(j+1)$. Also, $\mu_j = \mu$. Thus, the queueing system can be modeled as a birth-and-death process with the same parameters as the model of part (a). Consequently, $\{P_j\}$ is as in part (a). Furthermore, since the arrival process is Poisson, $\Pi_j = P_j$. We conclude that

(Chap. 3, Ex. 1 b)

$$\Pi_j = P_j = \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} \qquad (j = 0, 1, \ldots).$$

As in part (a),

$$a' = 1 - e^{-\lambda\tau}.$$

However, the offered load is

$$a = \lambda\tau.$$

Letting P denote the probability that an arbitrary arrival does not receive service,

$$P = \sum_{j=0}^{\infty} \frac{j}{j+1} \Pi_j = \sum_{j=0}^{\infty} \left(1 - \frac{1}{j+1}\right) P_j = 1 - \frac{1}{\lambda\tau} \sum_{j=0}^{\infty} \frac{(\lambda\tau)^{j+1}}{(j+1)!} e^{-\lambda\tau}$$

$$= 1 - \frac{1 - e^{-\lambda\tau}}{\lambda\tau} = 1 - \frac{a'}{a} = \frac{a - a'}{a}.$$

$\boxed{c}$ As in the models of parts (a) and (b), $\lambda_j / \mu_{j+1} = \lambda / [(j+1)\mu]$, so the state distribution $\{P_j\}$ is the same in those cases. Furthermore, since the arrival process is Poisson, $\Pi_j = P_j$. Hence,

$$\Pi_j = P_j = \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} \qquad (j = 0, 1, \ldots).$$

Also, as in parts (a) and (b), the carried load is

$$a' = 1 - e^{-\lambda\tau}.$$

The birth-and-death process will not be affected by preemption coupled with service in reverse order of arrival. In this case, a customer who arrives in state j will be served at rate μ_{j+1} when in service, and his mean service time, which is not affected by preemption, is μ_{j+1}^{-1}. Thus the overall mean service time equals

$$\overline{\mu} = \sum_{j=0}^{\infty} \Pi_j \mu_{j+1}^{-1} = \sum_{j=0}^{\infty} \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} \frac{1}{(j+1)\mu} = \frac{\mu^{-1}}{\lambda\tau} \sum_{j=0}^{\infty} \frac{(\lambda\tau)^{j+1}}{(j+1)!} e^{-\lambda\tau} = \frac{1}{\lambda}(1 - e^{-\lambda\tau})$$

The offered load is therefore $a = \lambda\overline{\mu} = 1 - e^{-\lambda\tau}$. Hence, $a = a'$, of course.

$\square$

Chapter 3, Exercise 2

'Customers arrive at a two-chair shoe-shine stand...'

$\boxed{a}$ $\lambda = 10,\ \mu = 10,\ s = 1,\ k = 1$

The corresponding birth-and-death model has: $\lambda_0 = \lambda_1 = \lambda = 10$, $\lambda_2 = 0$; $\mu_0 = 0$, $\mu_1 = \mu_2 = \mu = 10$. By (1.1),

$$P_0 = \left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\right)^{-1}, \quad P_1 = \frac{\lambda_0}{\mu_1}P_0, \quad P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0.$$

Hence,

$$(P_0, P_1, P_2) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

$\boxed{b}$ The mean number of customers served per hour is

$$\Delta = \mu_1 P_1 + \mu_2 P_2 = \frac{10}{3} + \frac{10}{3} = \underline{6.67}$$

$\boxed{c}$ $\lambda = 10,\ \mu = 10,\ s = 2,\ k = 0$

The corresponding birth-and-death model has: $\lambda_0 = \lambda_1 = \lambda = 10$, $\lambda_2 = 0$; $\mu_0 = 0$, $\mu_1 = \mu = 10$, $\mu_2 = 2\mu = 20$. Applying the above formulas we find

$$(P_0, P_1, P_2) = \left(\tfrac{4}{10}, \tfrac{4}{10}, \tfrac{2}{10}\right)$$

and

$$\Delta = \underline{8.00}.$$

Chapter 3, Exercise 3

'Derive (3.5) from the definition (1.9) and the probabilities (3.3)'

$$a' = \sum_{j=1}^{s} j P_j = \frac{\sum_{j=1}^{s} j\, a^j/j!}{\sum_{k=0}^{s} a^k/k!} = a\,\frac{\sum_{j=1}^{s} a^{j-1}/(j-1)!}{\sum_{k=0}^{s} a^k/k!}$$

$$= a\,\frac{\sum_{j=0}^{s-1} a^j/j!}{\sum_{k=0}^{s} a^k/k!} = a\left[1 - \frac{a^s/s!}{\sum_{k=0}^{s} a^k/k!}\right]. \qquad [a = \lambda/\mu]$$

By (3.4), then

$$a' = a[1 - B(s,a)] \qquad (3.5) \qquad \square$$

Chapter 3, Exercise 4

'Consider an Erlang loss system with 10 servers.'

The solution requires the evaluation of $B(s,a)$. Figures A-1 and A-2 of Appendix A provide the answers. Alternatively one can use a table of the cumulative Poisson distribution, since

$$B(s,a) = \frac{\frac{a^s}{s!} e^{-a}}{\sum_{k=0}^{s} \frac{a^k}{k!} e^{-a}},$$

and both numerator and denominator may be read off or easily calculated from a table with cumulative Poisson probabilities. We find

$$B(10,4.5) = \frac{0.0104}{0.9933} = 0.0105$$

(and $B(10,4.0) = 0.0053$). Accordingly we accept $a = 4.5$ as an approximate solution of $B(10,a) = 0.01$. We also find

$$B(16,9.0) = 0.0110, \qquad B(17,9.0) = 0.0058.$$

Thus, a doubling of the offered load does not necessitate a doubling of the number of servers, from 10 to 20, in order to prevent service degradation. Only 7 servers need be added to the system.

Chapter 3, Exercise 5

'An entrepreneur offers services...'

The offered load is $a = \lambda\tau = 4 * 1 = 4.0$. The hourly profit at operating cost c equals $H(s;c) = \lambda[1-B(s,a)]\,2.5 - sc = 10[1-B(s,4.0)] - sc$. Hence,

s	1	2	3	4	5	6	7	8	9	10
$B(s,4.0)$	.800	.615	.451	.311	.199	.117	.063	.030	.013	.005
$H(s;1.0)$	1.00	1.85	2.49	2.89	3.01	2.83	2.37	1.70	0.87	-0.05

Thus, at $c = 1.0$, the optimal number of servers is 5, and the corresponding profit rate equals 3.01. The break-even point for c is $c_0 = 2.0$: with $s = 1$ the entrepreneur will just break even; with $s > 1$ he will lose. $\square$

Chapter 3, Exercise 6

'Show that $B(s,a) = aB(s-1,a)/[s + aB(s-1,a)]$.'

By (3.4), for $s \geq 1$,

$$B(s,a) = \frac{a^s/s!}{\sum_{k=0}^{s} a^k/k!} = \frac{a}{s} \frac{a^{s-1}/(s-1)!}{\sum_{k=0}^{s-1} a^k/k!} \frac{\sum_{k=0}^{s-1} a^k/k!}{\sum_{k=0}^{s} a^k/k!}$$

$$= \frac{a}{s} B(s-1,a)[1 - B(s,a)],$$

where $B(0,a) \equiv 1$. Solving for $B(s,a)$ we derive

$$B(s,a) = \frac{aB(s-1,a)}{s + aB(s-1,a)} \qquad (s = 1,2,\ldots).$$

Chapter 3, Exercise 7

'Consider an Erlang loss system with retrials.'

No comment.

Chapter 3, Exercise 8

'Consider an equilibrium s-server Erlang loss system ...'

In the Erlang loss system the event {next arrival is blocked} will occur if and only if (a) the observer finds all servers busy, and (b) next arrival occurs before next service completion. Obviously, event (a) has probability $B(s,a)$ ($= P_s = \Pi_s$). Given (a), event (b) has probability $\lambda/(\lambda + s\mu)$, by Eq. (5.23) of Chapter 2, as time to next arrival and time to next service completion are independent exponential variables with parameters λ and $s\mu$, respectively. Hence

$$p = B(s,a)\frac{\lambda}{\lambda + s\mu} = \frac{a}{a+s}B(s,a).$$

The reason p is not equal to $B(s,a)$, as one might naively think, is that "next arrival" is not an arbitrary arrival. □

Chapter 3, Exercise 9

'The Erlang loss system as a semi-Markov process.'

We consider the s-server Erlang loss system with exponential service times, and let λ = arrival rate and μ = service rate.

a Clearly,

$$p_{ij} = \begin{cases} 0 & (|i-j| \neq 1), \\ \lambda/(\lambda+i\mu) & (0 \leq i \leq s-1,\ j=i+1), \\ i\mu/(\lambda+i\mu) & (1 \leq i \leq s-1,\ j=i-1), \\ 1 & (i=s,\ j=s-1). \end{cases} \qquad (6)$$

b Clearly,

$$m_i = \begin{cases} 1/(\lambda+i\mu) & (0 \leq i \leq s-1), \\ 1/s\mu & (i=s). \end{cases} \qquad (7)$$

c Substitution of Eq. (6) into Eq. (3) yields

$$P_0^* = \frac{\mu}{\lambda+\mu} P_1^* \qquad (s>1),$$

$$P_i^* = \frac{\lambda}{\lambda+(i-1)\mu} P_{i-1}^* + \frac{(i+1)\mu}{\lambda+(i+1)\mu} P_{i+1}^* \qquad (s>2,\ 1 \leq i \leq s-2),$$

$$P_{s-1}^* = \frac{\lambda}{\lambda+(s-2)\mu} + 1 \cdot P_s^* \qquad (s>1),$$

$$P_s^* = \frac{\lambda}{\lambda+(s-1)\mu} P_{s-1}^* .$$

By recursive solution we obtain

$$P_i^* = \begin{cases} \frac{\lambda+i\mu}{\lambda} \frac{(\lambda/\mu)^i}{i!} P_0^* & (0 \leq i \leq s-1), \\ \frac{(\lambda/\mu)^{s-1}}{(s-1)!} P_0^* & (i=s). \end{cases} \qquad (8)$$

Now, by (7) and (8),

$$m_i P_i^* = \frac{1}{\lambda} \frac{(\lambda/\mu)^i}{i!} P_0^* \qquad (0 \leq i \leq s). \qquad (9)$$

Inserting this expression into Eq. (5), with $k=s$, we derive Eq. (33):

$$P_j = \frac{(\lambda/\mu)^j/j!}{\sum_{i=0}^{s} (\lambda/\mu)^i/i!} \qquad (j=0,1,\dots,s) \qquad \square$$

Chapter 3, Exercise 10

'Two independent Poisson streams of traffic...'

Let the high priority stream parameters be λ_1 and τ_1, where $\lambda_1 = 20$ and $\tau_1 = 0.2$, and let the low priority stream parameters be λ_2 and τ_2. The two service time distributions may be general. For $s = 10$, the average overflow rate of high priority customers is known to be $\overline{\lambda}_1 = 2$. We wish to determine $a_2 = \lambda_2 \tau_2$.

The primary group serves two independent streams of Poisson traffic on a BCC basis. Therefore, as argued in the text, the primary system is an Erlang loss system with arrival rate $\lambda = \lambda_1 + \lambda_2$ and a mixed service time distribution with the mean $\tau = (\lambda_1/\lambda)\tau_1 + (\lambda_2/\lambda)\tau_2$. The total offered load is $a = \lambda\tau = \lambda_1\tau_1 + \lambda_2\tau_2 = 20 \times 0.2 + \lambda_2\tau_2 = 4 + a_2$. The percentage overflow of high priority customers clearly is $\overline{\lambda}_1/\lambda_1 = 2/20 \doteq 0.10$. The same percentage will overflow from each stream arriving at the primary group, so $B(s,a) = 0.10$. That is, $B(10,a) = 0.10$. Solving by use of the graph in Appendix A-I we find $a = 7.5$ ($B(10,7.5) = 0.0995$). Hence, $a_2 = a - a_1 = 7.5 - 4.0$. Thus,

$$a_2 = 3.5.$$

$\boxed{a}$ Denote by λ_2^*, a_2^* and a^* the new values of λ_2, a_2 and a. We have $\lambda_2^* = 2\lambda_2$. Hence, $a_2^* = \lambda_2^*\tau_2 = 2\lambda_2\tau_2 = 2a_2 = 7$, whereby $a^* = a_1 + a_2^* = 4 + 7 = 11$. It follows that the new overflow rate (average) of high priority customers will be

$$\overline{\lambda}_1^* = \lambda_1 B(s,a^*) = 20\, B(10,11) = 20 \times 0.260 = 5.2.$$

The factor of increase is

$$\frac{\overline{\lambda}_1^*}{\overline{\lambda}_1} = \frac{5.2}{2.0} = 2.6.$$

$\boxed{b}$ It is not permissible to design the backup group by use of Erlang's loss formula which assumes Poisson traffic. The overflow traffic is not Poisson. Disregarding this fact will lead to underestimation of the loss on the backup group, one would think. □

Chapter 3, Exercise 11

'Prove equation (3.12)'.

For all $t > 0$,

$$0 \leq t[1-H(t)] = \int_t^\infty t\,dH(x) \leq \int_t^\infty x\,dH(x).$$

Now, $\mu^{-1} = \int_0^\infty x\,dH(x) < \infty$ implies $\lim_{t\to\infty} \int_t^\infty x\,dH(x) = 0$. Hence, taking limits we obtain

$$0 \leq \lim_{t\to\infty} t[1-H(t)] \leq \lim_{t\to\infty} \int_t^\infty x\,dH(x) = 0.$$

Thus,

$$\lim_{t\to\infty} t[1-H(t)] = 0. \qquad (3.12)$$

Chapter 3, Exercise 12

'Blocked customers held.'

$\boxed{a}$ Each customer stays in the system (queue + service) for a time T that follows the sojourn time distribution H(x). Thus the queueing system may be modeled as an infinite server queue where the sojourn time is interpreted as a service time. It follows that $\{P_j(t)\}$ is the Poisson distribution

$$P_j(t) = \frac{[\lambda t p(t)]^j}{j!} e^{-\lambda t p(t)} \qquad (j = 0, 1, \dots). \qquad (3.11)$$

with

$$p(t) = 1 - H(t) + \int_0^t \frac{x}{t}\,dH(x), \qquad (3.9)$$

where now H(x) is the sojourn time distribution function.

$\boxed{b}$ Assume that T has the exponential distribution with mean μ^{-1}. Customers may defect before reaching the server. For a customer who does enter service, the remaining sojourn time (= service time) will, by the Markov property, also be exponentially distributed with mean μ^{-1}.

(Chap. 3, Ex. 12 c)

$\boxed{c}$ As in part (b), let T follow the exponential distribution. Let δ denote the mean defection rate (from queue). In state $j>s$ the defection rate is $(j-s)\mu$. Hence

$$\delta = \sum_{j=s+1}^{\infty} (j-s)\mu P_j .$$

By (3.8),

$$P_j = \frac{(\lambda/\mu)^j}{j!} e^{-\frac{\lambda}{\mu}} \quad (j = 0,1,2,\ldots).$$

μ^{-1} is both mean sojourn time and mean service time (in the normal sense), so $a = \lambda/\mu$ is the offered load. Thus

$$\begin{aligned} q = \frac{\delta}{\lambda} &= \frac{1}{\lambda} \sum_{j=s+1}^{\infty} (j-s)\mu \frac{a^j}{j!} e^{-a} \\ &= \sum_{j=s+1}^{\infty} \frac{a^{j-1}}{(j-1)!} e^{-a} - \frac{s}{a} \sum_{j=s+1}^{\infty} \frac{a^j}{j!} e^{-a} \\ &= \sum_{j=s}^{\infty} \frac{a^j}{j!} e^{-a} - \frac{s}{a} \sum_{j=s+1}^{\infty} \frac{a^j}{j!} e^{-a}. \end{aligned}$$

That is,

$$q = P(s,a) - \frac{s}{a} P(s+1,a).$$

Per unit time $\lambda[1-q]$ $(=\lambda-\delta)$ will enter service. The mean service time equals μ^{-1}. It follows that the carried load equals $a' = \lambda[1-q]\mu^{-1} = a[1-q]$. Thus, $q = 1 - a'/a$.

Chapter 3, Exercise 13

'Suppose that a company with a private telephone network...'

Let s_1 = number of flat rate trunks, s_2 = number of measured rate trunks. Assume an ordered hunt such that a call will be carried by a flat rate trunk whenever possible. Evidently, this policy will minimize the relevant costs. The priority within the two classes of trunks is immaterial.

The associated hourly cost is

$$H(s_1, s_2) = 14 s_1 + 30 \sum_{j=s_1+1}^{s_1+s_2} \tilde{P}_j ,$$

(Chap. 3, Ex. 13)

where

$$\tilde{p}_j = a\,[B(j-1,a) - B(j,a)], \qquad (3.18)$$

with $a = 2$ erlangs.

If $14 > 30\tilde{p}_1$, let $j^* = 0$. If $14 \leqq 30\tilde{p}_1$, let j^* be the maximal j such that $14 \leqq 30\tilde{p}_j$. By (3.18), $\tilde{p}_2 = 2(0.6667 - 0.4000) = 0.5333$ and $\tilde{p}_3 = 2(0.4000 - 0.2105) = 0.3790$. Hence, $14 < 30\tilde{p}_2 = 16.00$, but $14 > 30\tilde{p}_3 = 11.37$. As $\tilde{p}_1 > \tilde{p}_2 > \cdots$, obviously $j^* = 2$.

Considering the cost function $H(s_1, s_2)$ and the relations $\tilde{p}_1 > \tilde{p}_2 > \cdots$,

$$s_1^* = \min(j^*, s) \qquad (*)$$

is the optimal number of flat rate trunks out of a total of s $(= s_1 + s_2)$ trunks.

It is a requirement that $B(s_1+s_2, 2) \leqq 0.02$. We have $B(5,2) = 0.0367$ and $B(6,2) = 0.0121$, and since the cost structure does not explicitly account for blocking costs, $s_1 + s_2 = 6$ is the optimal number of trunks. Hence, by (*),

$$s_1^* = \min(2, 6) = \underline{2}$$

is the optimal number of trunks, and the associated cost is

$$\begin{aligned} H(2,4) &= 14 \cdot 2 + 30(\tilde{p}_3 + \tilde{p}_4 + \tilde{p}_5 + \tilde{p}_6) \\ &= 28 + 30a[B(2,a) - B(6,a)] \qquad [\text{by } (3.18)] \\ &= 28 + 60(0.4000 - 0.0121) \\ &= \underline{51.27}. \end{aligned}$$

Given $s_1 + s_2 = 6$, the direct approach is to calculate $H(s_1, 6-s_1)$ $(= 14s_1 + 60(B(s_1,2) - B(6,2)))$ for $s_1 = 0, 1, \ldots, 6$. The result is:

s_1	0	1	2	3	4	5	6
$H(s_1, 6-s_1)$	59.27	53.28	51.27	53.40	60.99	71.48	84.00

Again, $s_1^* = 2$.

□

Chapter 3, Exercise 14

'Prove that in an Erlang loss system with ordered hunt...'

Suppose there are s servers. Consider an arbitrary customer; denote by A_j the event that on arrival he finds the first j servers busy, and let E_j denote the event that the customer will be served by server j, meaning that the first $j-1$ servers are busy, whereas server j is free. Obviously, $A_j \subset A_{j-1}$ and $E_j = A_{j-1} - A_j$. Hence, $P\{E_j\} = P\{A_{j-1}\} - P\{A_j\}$ $(j=1,\ldots,s)$. With ordered hunt the first j servers, $j \leq s$, function as an Erlang loss system, so $P\{A_j\} = B(j,a)$. Thus,

$$P\{E_j\} = B(j-1,a) - B(j,a) \qquad (j=1,\ldots,s).$$

By (3.18),

$$\frac{\tilde{p}_j}{a} = B(j-1,a) - B(j,a) \qquad (j=1,\ldots,s).$$

It follows that

$$P\{E_j\} = \frac{\tilde{p}_j}{a} \qquad (j=1,\ldots,s).$$

Chapter 3, Exercise 15

'Prove that the variance v of the Erlang loss distribution...'

We shall demonstrate that the variance of the state variable J with distribution

$$P\{J=j\} = P_j = \frac{a^j/j!}{\sum_{k=0}^{s} a^k/k!} \qquad (j=0,1,\ldots,s) \qquad (3.3)$$

may be expressed as $V(J) = v = a'(1-\tilde{p}_s)$.

First we prove the formula in the simple case $s=1$. By (3.5),

$$E(J) = \sum_{j=0}^{1} jP_j = a' = a - aB(1,a).$$

J is a zero-one variable, so that $E(J^2) = E(J)$. Hence

$$E(J^2) = a - aB(1,a).$$

(Chap. 3, Ex. 15)

Hence,

$$V(J) = E(J^2) - E^2(J) = a[1-B(1,a)](1-a[1-B(1,a)]).$$

By (3.5), $a' = a[1-B(1,a)]$, and by (3.18), $\tilde{p}_1 = a[1-B(1,a)]$. Thus

$$V(J) = v = a'(1-\tilde{p}_s) \qquad (s=1).$$

Now consider the case $s \geq 2$. To begin, we express the variance $V(J)$ in terms of s, a and $B(s,a)$:

$$E(J) = \textstyle\sum_{j=0}^{s} j P_j = a' = a - a\,B(s,a). \qquad \text{[by (3.5)]}$$

$$E(J(J-1)) = \textstyle\sum_{j=0}^{s} j(j-1)P_j = \sum_{j=2}^{s} j(j-1)P_j = \dfrac{a^2 \sum_{j=2}^{s} a^{j-2}/(j-2)!}{\sum_{j=0}^{s} a^j/j!}$$

$$= a^2\Big(1 - (1+\tfrac{s}{a})\dfrac{a^s/s!}{\sum_{j=0}^{s} a^j/j!}\Big) = a^2 - a^2 B(s,a) - as\,B(s,a).$$

$$E(J^2) = E(J(J-1)) + E(J) = a^2 - a^2 B(s,a) - as\,B(s,a) + a - a\,B(s,a).$$

$$E^2(J) = (a - a\,B(s,a))^2 = a^2 - 2a^2 B(s,a) + a^2 B^2(s,a).$$

Hence,

$$V(J) = E(J^2) - E^2(J) = a - a\,B(s,a) - as\,B(s,a) + a^2 B(s,a) - a^2 B^2(s,a),$$

which can be rewritten

$$V(J) = a[1-B(s,a)]\Big(1 - \frac{s\,B(s,a)}{1-B(s,a)} + a\,B(s,a)\Big).$$

If the equation of Exercise 6 is solved w.r.t. $B(s-1,a)$ we find

$$\frac{s\,B(s,a)}{1-B(s,a)} = a\,B(s-1,a).$$

Thus,

$$V(J) = a[1-B(s,a)]\big(1 - a[B(s-1,a) - B(s,a)]\big).$$

Finally, using Equations (3.5) and (3.18) we obtain

$$V(J) = v = a'(1-\tilde{p}_s) \qquad (s \geq 1).$$

$\square$

Chapter 3, Exercise 16

'a. Show that, for every integer $s > a$, $C(s,a) = \ldots$'

$\boxed{a}$ A rewriting of (4.8) gives

$$C(s,a) = \frac{\frac{s}{s-a}\frac{a^s}{s!}}{\sum_{k=0}^{s}\frac{a^k}{k!} + \frac{a}{s-a}\frac{a^s}{s!}} \qquad (s > a).$$

Dividing numerator and denominator by $\sum_{k=0}^{s} a^k/k!$ and introducing $B(s,a) = (a^s/s!)/\sum_{k=0}^{s} a^k/k!$ we easily derive

$$C(s,a) = \frac{s\,B(s,a)}{s - a(1 - B(s,a))} \qquad (s > a). \qquad (1)$$

$\boxed{b}$ Another rewriting of (4.8) gives

$$C(s,a) = \frac{\frac{a}{s-a}\frac{a^{s-1}}{(s-1)!}}{\sum_{k=0}^{s-1}\frac{a^k}{k!} + \frac{a}{s-a}\frac{a^{s-1}}{(s-1)!}} \quad (s > a).$$

Dividing numerator and denominator by $\sum_{k=0}^{s-1} a^k/k!$ and introducing $B(s-1,a) = (a^{s-1}/(s-1)!)/\sum_{k=0}^{s-1} a^k/k!$ we obtain

$$C(s,a) = \frac{1}{1 + (s-a)[a\,B(s-1,a)]^{-1}} \qquad (s > a). \qquad (2)$$

$\boxed{c}$ By (1), for $s-1 > a$, that is, for $s > a+1$,

$$C(s-1,a) = \frac{(s-1)\,B(s-1,a)}{(s-1) - a(1 - B(s-1,a))} \qquad (s > a+1).$$

Solving for $B(s-1,a)$ leads to

$$B(s-1,a) = \frac{(s-1-a)\,C(s-1,a)}{s-1-a\,C(s-1,a)} \qquad (s > a+1).$$

Insertion of the above expression into (2) results in

$$C(s,a) = \frac{1}{1 + \left(\frac{s-a}{a}\right)\frac{s-1-a\,C(s-1,a)}{(s-1-a)\,C(s-1,a)}} \qquad (s > a+1). \qquad (3)$$

□

Chapter 3, Exercise 17

'Review and reconsider Exercises 4 and 5 of Chapter 1.'

The equilibrium state probabilities $\{P_j\}$ derived for the delay system in Exercise 4 and for the loss-delay system in Exercise 5, hold for Poisson arrivals and exponential service times. This may be proved rigorously by modeling the systems as birth-and-death processes, and then applying Eq. (1.1).

For the loss-delay system of Exercise 5 of Chapter 1, let s = number of servers, n = waiting room size, λ = arrival rate, μ = service rate, offered load $a = \lambda/\mu$. It was found that

$$P_j = \begin{cases} \dfrac{a^j}{j!} P_0 & (j = 1, 2, \ldots, s-1), \\ \dfrac{a^j}{s!\, s^{j-s}} P_0 & (j = s, \ldots, s+n), \end{cases} \qquad (*)$$

where $P_0 = \left(\sum_{k=0}^{s-1} a^k/k! + [a^s/s!] \sum_{i=0}^{n} (a/s)^i \right)^{-1}$, or,

$$P_0 = \left(\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{s!} \, \frac{1 - (\frac{a}{s})^{n+1}}{1 - \frac{a}{s}} \right)^{-1}. \qquad (**)$$

Denote by P_L, P_W, P_S, the equilibrium probabilities of being lost (denied service), having to wait in queue, and getting served immediately. Note, $\Pi_j = P_j$, since the arrival process is Poisson. Hence,

$$P_L = \Pi_{s+n} = P_{s+n},$$

$$P_W = \sum_{j=s}^{s+n-1} \Pi_j = \sum_{j=s}^{s+n-1} P_j \qquad (n \geqq 1),$$

$$P_S = \sum_{j=0}^{s-1} \Pi_j = \sum_{j=0}^{s-1} P_j .$$

By $(*)$,

$$P_L = \frac{a^{s+n}}{s!\, s^n} P_0,$$

$$P_W = \frac{a^s}{s!} \, \frac{1 - (a/s)^n}{1 - (a/s)} P_0 \qquad (n \geqq 1),$$

$$P_S = \sum_{j=0}^{s-1} \frac{a^j}{j!} P_0,$$

with P_0 given by $(**)$. □

Chapter 3, Exercise 18

'Is the analysis leading to (4.12) valid for...'

The answer is no. The reason is that the mean idle period, which is the mean residual interarrival time at the end of the busy period, is not, in general, equal to the mean interarrival time λ^{-1} as in the case of Poisson arrivals, i.e. exponentially distributed interarrival times.

Chapter 3, Exercise 19

'Consider again the premise of Exercise 13. Now, however,...'

The subject is an Erlang delay system with $s=4$ trunks and $a=2$. Let s_1 = number of flat-rate trunks, and let $s-s_1 = 4-s_1$ = number of measured-rate trunks. s_1 must be set to minimize the ordered hunt hourly cost, assuming that flat-rate trunks have priority,

$$H(s_1) = 14 s_1 + 30 \sum_{j=s_1+1}^{4} P_j , \qquad (*)$$

where

$$P_j = \tilde{P}_j[1-\rho C(s,a)] + \rho C(s,a), \qquad (4.16)$$

and

$$\tilde{P}_j = a[B(j-1,a) - B(j,a)]. \qquad (3.18)$$

Here, $\rho = a/s = 0.5$, and $C(s,a) = C(4,2) = 0.1739$ according to tables of the Erlang delay formula (4.8), see Fig. A-3, Appendix A. $\tilde{P}_j$ is determined using tables of the Erlang loss formula (3.4). We find

j	1	2	3	4
$\tilde{P}_j$	.6667	.5334	.3790	.2306
P_j	.6957	.5740	.4330	.2975

Substitution of the P_j's into (*) yields

s_1	0	1	2	3	4
$H(s_1)$	60.01	53.14	49.92	50.93	56.00

Best choice therefore is $s_1^* = 2$, and $H(s_1^*) = 49.92$. □

Chapter 3, Exercise 20

'Prove that for an s-server Erlang delay system...'

Special version of theorem

Let $\{P_j^\circ\}$ be the equilibrium state probabilities of an Erlang loss system, and let $\{P_j^*\}$ be the equilibrium state probabilities of an Erlang delay system. Suppose the systems have the same number of servers s and identical parameters λ and μ. In the delay system, let Q_j^* be defined as the conditional probability of state j, given $j \le s$, that is, $Q_j^* = P_j^*/\sum_{k=0}^{s} P_k^*$ $(j = 0,1,\dots,s)$. Then $P_j^\circ = Q_j^*$ $(j = 0,1,\dots,s)$.

Proof. By (3.3),

$$P_j^\circ = \frac{(\lambda/\mu)^j/j!}{\sum_{k=0}^{s}(\lambda/\mu)^k/k!} \qquad (j = 0,1,\dots,s).$$

By (4.4),

$$P_j^* = \frac{(\lambda/\mu)^j}{j!} P_0^* \qquad (j = 0,1,\dots,s).$$

Assume that an equilibrium distribution exists, so that $P_0^* > 0$. Then

$$Q_j^* = \frac{P_j^*}{\sum_{k=0}^{s} P_k^*} = \frac{(\lambda/\mu)^j/j!}{\sum_{k=0}^{s}(\lambda/\mu)^k/k!} \qquad (j = 0,1,\dots,s).$$

Thus $P_j^\circ = Q_j^*$ for $j = 0,1,\dots,s$.

General version of theorem

Consider two birth-and-death processes with parameters $(\{\lambda_j^\circ\},\{\mu_j^\circ\})$ and $(\{\lambda_j^*\},\{\mu_j^*\})$. Assume equilibrium state distributions $\{P_j^\circ\}$ and $\{P_j^*\}$ exist and $P_0^\circ > 0$, $P_0^* > 0$. Assume $\lambda_j^\circ = \lambda_j^* = \lambda_j$ for $j = 0,\dots,s-1$ and $\mu_j^\circ = \mu_j^* = \mu_j$ for $j = 1,\dots,s$, for some $s \ge 1$. Let $Q_j^\circ = P_j^\circ/\sum_{k=0}^{s} P_k^\circ$ and $Q_j^* = P_j^*/\sum_{k=0}^{s} P_k^*$ $(j = 0,1,\dots,s)$ be the conditional probability of state j, given $j \le s$. Then $Q_j^\circ = Q_j^*$ for $j = 0,1,\dots,s$.

Proof. The result follows easily from the fact that $P_j^\circ/P_0^\circ = P_j^*/P_0^* = (\lambda_0\lambda_1\cdots\lambda_{j-1})/(\mu_1\mu_2\cdots\mu_j)$ for $j = 1,\dots,s$. Observe, in the special case above $Q_j^\circ = P_j^\circ$.

$\square$

Chapter 3, Exercise 21

'Reconsider Ex. 14 with "E. loss system" replaced by "E. delay system."

It will be shown that the statement made in Exercise 14 holds true also with "Erlang loss system" replaced by "Erlang delay system."

We consider an Erlang delay system with s servers and ordered hunt. Let X be an arbitrary customer. Let k be the state of the system when X arrives. Let $D = \{X \text{ is delayed}\}$, $\overline{D} = \{X \text{ is not delayed}\}$, $E_j = \{X \text{ is served by server } j\}$. Then,

$$P\{E_j\} = P\{E_j, D\} + P\{E_j, \overline{D}\}.$$

First we calculate $P\{E_j, D\}$. Write $P\{E_j, D\} = P\{E_j | D\} P\{D\}$. Given Poisson arrivals, $P\{D\} = \sum_{k=s}^{\infty} P_k = C(s,a)$, and given exponential service times, $P\{E_j | D\} = 1/s$. Thus

$$P\{E_j, D\} = \frac{1}{s} C(s,a).$$

Next we calculate $P\{E_j, \overline{D}\}$. Observe, $\{E_j, \overline{D}\}$ is equivalent to $\{k \leq s, E_j, \overline{D}\}$. With Poisson traffic, $P\{k \leq s\} = \sum_{k=0}^{s} P_k = 1 - \rho C(s,a)$, and conditional on $k \leq s$ the probability of service by server j without delay is $\tilde{p}_j / a$, according to Exercise 14, since the system functions like an Erlang loss system when $k \leq s$. Hence,

$$\begin{aligned} P\{E_j, \overline{D}\} &= P\{k \leq s, E_j, \overline{D}\} = P\{E_j, \overline{D} | k \leq s\} P\{k \leq s\}, \\ &= \frac{\tilde{p}_j}{a} [1 - \rho C(s,a)]. \end{aligned}$$

It follows that

$$P\{E_j\} = \frac{\tilde{p}_j [1 - \rho C(s,a)] + \rho C(s,a)}{a}$$

where $\tilde{p}_j = a[B(j-1,a) - B(j,a)]$. By (4.16) the numerator equals the load p_j carried by the j'th ordered server. Hence,

$$P\{E_j\} = \frac{p_j}{a} \quad (j = 1, 2, \ldots, s).$$

This result might have been easily derived by employing Little's theorem, $L = \lambda W$ (see Sec. 5.2), by which $p_j = \lambda P\{E_j\} \mu^{-1}$.

□

Chapter 3, Exercise 22

'Repeat Exercise 5 with "Erlang loss system" replaced by...'

We consider an Erlang delay system with $\lambda = 4$ and $\mu = 1$. Then the offered load is $a = \lambda\tau = \lambda\mu^{-1} = 4$. The objective function is $H(s;c) = \lambda 2.5 - \lambda P\{W > 0.5\} \cdot 10.0 - sc$. Thus

$$H(s;c) = 10 - 40\,P\{W > 0.5\} - sc.$$

By (4.25), $P\{W > t\} = C(s,a)\,e^{-(s\mu-\lambda)t}$. Thus

$$P\{W > 0.5\} = C(s,4)\,e^{-(s-4)0.5}.$$

It follows that

	5	6	7	8	9	10
$C(s,4)$	.5541	.2848	.1351	.0590	.0238	.0088
$e^{-(s-4)0.5}$	.6065	.3679	.2231	.1353	.0821	.0498
$P\{W>0.5\}$	.3361	.1048	.0301	.0080	.0020	.0004
$H(s;1.0)$	−8.44	−0.19	1.80	1.68	0.92	−0.01

Thus, at $c = 1.00$, the optimal number of servers is 7, and the corresponding profit rate equals 1.80. The break-even point for c is $c_0 = 1.00 + 1.80/7 = 1.26$. Given this operating cost, the entrepreneur will break even for $s = 7$, but will have a negative profit rate for $s \neq 7$.

In case the entrepreneur may select any customer from the queue, the profit will be maximized, for any s, if the customer selected is the one who has waited the longest, but less than ½ hr.

Chapter 3, Exercise 23

'Consider a 10-server Erlang delay system that handles...'

BCD	s	λ	μ^{-1}	$a = \lambda\mu^{-1}$	$C(s,a)$	$E(W\mid W>0) = \mu^{-1}/(s-a)$
case 0	10	λ_0	μ_0^{-1}	6	.1013	$W_0 = \mu_0^{-1}/4$
case 1	10	$(1+\frac{1}{3})\lambda_0$	μ_0^{-1}	8	.4092	$2W_0$
case 2	10	λ_0	$(1+\frac{1}{3})\mu_0^{-1}$	8	.4092	$2(1+\frac{1}{3})W_0$

(Chap. 3, Ex. 23)

Note, by (4.26), $E(W|W>0) = 1/[(1-\rho)s\mu]$, where $\rho = a/s$. Thus, $E(W|W>0) = \mu^{-1}/(s-a)$.

The lesson is that $C(s,a)$ depends on s and a, whereas the conditional mean wait $E(W|W>0)$ depends on s, a and μ. The response to a 1/3 increase in λ is a $100(0.4092-0.1013)/0.1013 = 304\%$ increase in $C(s,a)$ and a 100% increase in $E(W|W>0)$. A 1/3 increase in μ^{-1}, resulting in the same a, also leads to a 304% increase in $C(s,a)$, but the increase in $E(W|W>0)$ will be 167%.

Chapter 3, Exercise 24

'In an Erlang delay system with service in order of arrival...'

By (4.24), $P\{W>t|W>0\} = e^{-(1-\rho)s\mu t}$, and by (4.26), $E(W|W>0) = \frac{1}{(1-\rho)s\mu}$. Hence,

$$P\{W > E(W|W>0)|W>0\} = e^{-(1-\rho)s\mu\frac{1}{(1-\rho)s\mu}} = e^{-1} = 0.3679.$$

Chapter 3, Exercise 25

'Consider a telephone system in which the central office...'

In an Erlang delay system with service in order of arrival the waiting time distribution for blocked customers is the exponential distribution $P\{W>t|W>0\} = e^{-(1-\rho)s\mu t}$, see (4.24). Hence, if a customer has waited 30 sec., his remaining waiting time will still be exponentially distributed with mean $[(1-\rho)s\mu]^{-1}$.

One thing the customer should not do after waiting 30 sec. is to put down the receiver and try again immediately. If he does that and waits until he gets through to the server, he will increase the waiting time by an expected 30ρ secs due to those customers who, thanks to his rash act, got ahead of him in the waiting line.

A better choice is to hang up and make another call $T>0$ secs. later, waiting until served. The associated expected waiting time will converge to $C(s,a)/[(1-\rho)s\mu]$ as $T\to\infty$. As the limiting value is less than $1/[(1-\rho)s\mu]$, the customer may be better off, everything considered, calling later. □

Chapter 3, Exercise 26

'Show that in the Erlang delay system...'

$$E(W^2) = [1-C(s,a)]\,E(W^2|W=0) + C(s,a)\,E(W^2|W>0)$$

$$= C(s,a)\,E(W^2|W>0).$$

By (4.24), with order-of-arrival service the waiting time for blocked customers will be exponentially distributed with parameter $(1-\rho)s\mu$. Hence, $E(W^2|W>0) = 2/[(1-\rho)s\mu]^2$, so that

$$E(W^2) = \frac{2C(s,a)}{(s\mu)^2(1-\rho)^2}.$$

By (4.27),

$$E^2(W) = \frac{C^2(s,a)}{(s\mu)^2(1-\rho)^2}.$$

The variance is derived by substitution of these two expressions into $V(W) = E(W^2) - E^2(W)$. The result is

$$V(W) = \frac{1-(1-C(s,a))^2}{(s\mu)^2(1-\rho)^2}.$$

Chapter 3, Exercise 27

'Let W be the waiting time and T the sojourn time...'

$s=1$. Hence, by (4.4) and (4.5), $P_j = (1-a)a^j$. With Poisson arrivals $\Pi_j = P_j$, so the probability that an arbitrary customer finds j present in the system is

$$\Pi_j = (1-a)a^j \quad (j=0,1,\ldots). \tag{1}$$

The probability that he will observe j in the queue, given that the server is occupied is $P\{Q=j|W>0\} = \Pi_{j+1}/\sum_{k=1}^{\infty}\Pi_k = (1-a)a^{j+1}/a$. Hence, see also (4.23),

$$P\{Q=j\,|\,W>0\} = (1-a)a^j \quad (j=0,1,\ldots). \tag{2}$$

Now assume order-of-arrival service. The sojourn time will be the sum of $j+1$ exponential service times where the probability distribution of j (and therefore of $j+1$) is given by (1). The conditional waiting time is the sum of $j+1$ exponential service times where the probability distribution of j (and therefore of $j+1$) is given by (2).

The two probability distributions (1) and (2), are identical. Consequently, the sojourn time and the conditional waiting time follow the same distribution in this case, namely

$$P\{T>t\} = P\{W>t \mid W>0\} = e^{-(1-a)\mu t}$$

according to eq. (4.24).

Chapter 3, Exercise 28

'a. Consider an Erlang delay system, and denote by L...'

Suppose $a<s$. For convenience, let L_q and W_q denote the mean queue length and mean waiting time, resp., and let L_s and W_s denote the mean number of customers in the system and mean sojourn time, resp.

$\boxed{a}$ By (4.4) and (4.7),

$$L_q = \sum_{j=s}^{\infty}(j-s)P_j = \sum_{j=s}^{\infty}(j-s)\frac{a^j}{s!\,s^{j-s}}P_0$$
$$= \frac{a^s}{s!(1-a/s)}P_0\sum_{k=0}^{\infty}k\left(\frac{a}{s}\right)^k\left(1-\frac{a}{s}\right) = C(s,a)\sum_{k=0}^{\infty}k\left(\frac{a}{s}\right)^k\left(1-\frac{a}{s}\right).$$

The mean of a geometric distribution with parameter $p = 1-\frac{a}{s}$ is $(1-p)/p = \frac{a}{s}/(1-\frac{a}{s})$. Hence, $\sum_{k=0}^{\infty} k(\frac{a}{s})^k(1-\frac{a}{s}) = \frac{a}{s}/(1-\frac{a}{s})$. Substitution of this expression, and a rewriting, yield

$$L_q = \lambda\frac{C(s,a)}{(1-\rho)s\mu}.$$

Finally, by (4.27),

$$L_q = \lambda W_q.$$

$\boxed{b}$ Clearly $L_s = L_q + \frac{\lambda}{\mu}$ and $W_s = W_q + \frac{1}{\mu}$. Hence, using the relation $L_q = \lambda W_q$ it follows that

$$L_s = \lambda W_s.$$

□

Chapter 3, Exercise 29

'Prove that in an Erlang delay system with order-of-arrival service...'

Let P be the probability that a blocked customer will still be in the queue when next arrival takes place. Let r_j be the probability that a blocked customer, who joins the queue when $Q=j$ customers are waiting, will still be in the queue at next arrival epoch. By the theorem of total probability,

$$P = \sum_{j=0}^{\infty} r_j P\{Q=j \mid W>0\}.$$

The arrival rate is λ and, as long as all servers are busy, the service completion rate is $s\mu$. Therefore, by (5.23) of Chapter 2, $s\mu/(s\mu+\lambda)$ is the probability, in all-busy states, that next event will be a service completion rather than an arrival. Since, with service in order of arrival, the blocked customer will get into service before next arrival if and only if at least $j+1$ service completions occur before any arrival,

$$r_j = 1 - \left(\frac{s\mu}{s\mu+\lambda}\right)^{j+1} = 1 - \frac{1}{(1+\rho)^{j+1}}.$$

By (4.23), $P\{Q=j \mid W>0\} = (1-\rho)\rho^j$ if $\rho<1$. Hence, if $\rho<1$,

$$P = \sum_{j=0}^{\infty}\left(1-\frac{1}{(1+\rho)^{j+1}}\right)(1-\rho)\rho^j = 1 - \frac{1-\rho}{1+\rho}\sum_{j=0}^{\infty}\left(\frac{\rho}{1+\rho}\right)^j = 1 - \frac{1-\rho}{1+\rho}\,\frac{1}{1-[\rho/(1+\rho)]}.$$

Hence, $P=\rho$ as asserted.

Chapter 3, Exercise 30

'Let N be the number of customers found by an arrival...'

Evidently, eq. (4.19), $P\{W>t \mid W>0\} = \sum_{j=0}^{\infty} P\{W>t \mid N=s+j\}P\{Q=j \mid W>0\}$, holds for any Erlang delay system regardless of queue discipline. By definition, $P\{Q=j \mid W>0\} = P\{N=s+j\}/\sum_{k=0}^{\infty} P\{N=s+k\}$. For nonbiased queue disciplines $\{N(t)\}$ is a birth-and-death process, independent of the discipline. Hence $P\{N=k\}$, and therefore $P\{Q=j \mid W>0\}$, are the same for all nonbiased q.d. By (4.23), for order-of-arrival service $P\{Q=j \mid W>0\} = (1-\rho)\rho^j$ $(j=0,1,\ldots)$. It follows that for all nonbiased q.d. we have $P\{Q=j \mid W>0\} = (1-\rho)\rho^j$ $(j=0,1,\ldots)$. Substitution into (4.19) shows that

$$P\{W>t \mid W>0\} = (1-\rho)\sum_{j=0}^{\infty} \rho^j P\{W>t \mid N=s+j\},$$

for all nonbiased queue disciplines. $\square$

Chapter 3, Exercise 31

'Consider the differential-difference equations...

$$\frac{d}{dt}F_j(t) = cF_{j-1}(t) - cF_j(t) \qquad [t \geqq 0;\ j = 0, 1, \dots;\ F_{-1}(t) = 0] \qquad (1)$$

where c is an arbitrary constant. Define

$$F(x,t) = \sum_{j=0}^{\infty} F_j(t)x^j. \qquad (2)$$

$\boxed{a}$

$$\frac{d}{dt}F_j(t)x^j = cF_{j-1}(t)x^j - cF_j(t)x^j \qquad (j = 0, 1, \dots),$$

$$\sum_{j=0}^{\infty}\frac{d}{dt}F_j(t)x^j = cx\sum_{j=1}^{\infty}F_{j-1}(t)x^{j-1} - c\sum_{j=0}^{\infty}F_j(t)x^j,$$

$$\frac{d}{dt}\sum_{j=0}^{\infty}F_j(t)x^j = cx\sum_{j=0}^{\infty}F_j(t)x^j - c\sum_{j=0}^{\infty}F_j(t)x^j,$$

$$\frac{\partial}{\partial t}F(x,t) = c(x-1)F(x,t). \qquad (3)$$

Hence, $F(x,t) = k(x)e^{-(1-x)ct}$, by which

$$F(x,t) = F(x,0)e^{-(1-x)ct}. \qquad (4)$$

$\boxed{b}$ In the case of a Poisson process Eq. (1) holds with $c = \lambda$ and $F_j(t) = P_j(t) = P\{N(t) = j\}$, according to Eq. (2.5) of Chapter 2. As $F_0(0) = 1$, clearly $F(x,0) = 1$, so that in this case $F(x,t) = e^{-(1-x)ct} = e^{-\lambda t(1-x)}$. By Eq. (4.3) of Chapter 2, this is the generating function of a Poisson distribution with parameter λt. It follows that the probability of j arrivals in $[0,t]$ equals

$$P_j(t) = \frac{(\lambda t)^j}{j!}e^{-\lambda t} \qquad (j = 0, 1, \dots).$$

$\boxed{c}$ As long as there are at least s customers in the system, the departure process is Poisson with parameter $s\mu$. Let $\tilde{P}_i(t)$ denote the probability of i departures within $[0,t]$, assuming that all servers are busy. By the usual argument,

$$\tilde{P}_i(t+h) = \tilde{P}_i(t)[1 - hs\mu] + \tilde{P}_{i-1}(t)hs\mu + o(h) \qquad (i = 0, 1, \dots).$$

(Chap. 3, Ex. 31 c)

Hence, with $\tilde{P}_{-1}(t) = 0$,

$$\sum_{i=0}^{j} \tilde{P}_i(t+h) = [1 - hs\mu] \sum_{i=0}^{j} \tilde{P}_i(t) + hs\mu \sum_{i=0}^{j-1} \tilde{P}_i(t) + o(h) \qquad (j = 0, 1, \dots)$$

Evidently, $W_j(t) = \sum_{i=0}^{j} \tilde{P}_i(t)$, so that

$$W_j(t+h) = [1 - hs\mu] W_j(t) + hs\mu W_{j-1}(t) + o(h) \qquad (j = 0, 1, \dots),$$

where $W_{-1}(t) = 0$. Hence,

$$\frac{d}{dt} W_j(t) = s\mu W_{j-1}(t) - s\mu W_j(t) \quad [t \geq 0;\ j = 0, 1, \dots;\ W_{-1}(t) = 0] \tag{5}$$

$\boxed{d}$ Eq. (5) has the same form as Eq. (1). Consequently, if we define

$$W(x,t) = \sum_{j=0}^{\infty} W_j(t) x^j, \tag{6}$$

then, by (2) and (4),

$$W(x,t) = W(x,0)\, e^{-(1-x)s\mu t}. \tag{7}$$

$\boxed{e}$ $W_j(0) = P\{W > 0 \mid N = j + s\} = 1$ for all j. Hence, for $x < 1$,

$$W(x,0) = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}. \tag{8}$$

$\boxed{f}$

$$\begin{aligned} P\{W > t \mid W > 0\} &= (1-\rho) \sum_{j=0}^{\infty} P\{W > t \mid N = s + j\} \rho^j && [\text{by Ex. 30}] \\ &= (1-\rho) \sum_{j=0}^{\infty} W_j(t) \rho^j && [\text{by def. of } W_j(t)] \\ &= (1-\rho)\, W(\rho, t). && [\text{by def. of } W(x,t)] \end{aligned} \tag{9}$$

$\boxed{g}$ Equations (7), (8), (9) together yield

$$\begin{aligned} P\{W > t \mid W > 0\} &= (1-\rho) \frac{1}{1-\rho}\, e^{-(1-\rho)s\mu t} \\ &= e^{-(1-\rho)s\mu t}. \end{aligned} \tag{10}$$

(Chap. 3, Ex. 31 h)

$\boxed{h}$ By Equations (6), (7) and (8),

$$\sum_{j=0}^{\infty} W_j(t)\, x^j = \frac{1}{1-x}\, e^{-s\mu t} e^{s\mu t x}.$$

Thus,

$$\sum_{j=0}^{\infty} W_j(t)\, x^j = \Big(\sum_{i=0}^{\infty} x^i\Big)\, e^{-s\mu t}\Big(\sum_{i=0}^{\infty} \frac{(s\mu t)^i}{i!}\, x^i\Big),$$

$$\sum_{j=0}^{\infty} W_j(t)\, x^j = \sum_{j=0}^{\infty}\Big(\sum_{k=0}^{j} \frac{(s\mu t)^k}{k!}\, e^{-s\mu t}\Big)\, x^j.$$

Equating coefficients of x^j on left- and right-hand sides yields

$$W_j(t) = \sum_{k=0}^{j} \frac{(s\mu t)^k}{k!}\, e^{-s\mu t}. \qquad (11)$$

Chapter 3, Exercise 32

'Service in random order.' – cf Ex. 28 and 36 of Chap. 5

$$W_j(t) = P\{W > t \mid N = s + j\}$$

$\boxed{a}$ Let the test customer arrive at $t = 0$. During the time interval $[0, h]$ one of the following mutually exclusive events will occur: (1) The test customer departs from queue, (2) A customer arrives, (3) A customer other than the test customer departs from queue, (4) Neither arrival nor departure from queue (system) take place; (5) Two or more arrivals or departures occur.

Event 1 precludes the possibility that the test customer will be present in the queue at time $h+t$, and event 5 has probability $o(h)$. Disregarding terms of order $o(h)$, events 2, 3 and 4 have probability λh, $(j/j+1)s\mu h$ and $1-(\lambda+s\mu)h$, respectively. Hence, by the theorem of total probability,

$$W_j(h+t) = \lambda h\, W_{j+1}(t) + \frac{j}{j+1} s\mu h\, W_{j-1}(t) + [1-(\lambda+s\mu)h]\, W_j(t) + o(h), \qquad (1)$$
$$[j = 0, 1, \ldots;\ W_{-1}(t) = 0].$$

Hence,

$$\frac{d}{dt} W_j(t) = \lambda W_{j+1}(t) + \frac{j}{j+1} s\mu\, W_{j-1}(t) - (\lambda + s\mu) W_j(t) \qquad [j = 0, 1, \ldots;\ W_{-1}(t) = 0], \quad (2)$$

where $W_j(0) = 1 \quad (j = 0, 1, \ldots)$.

(Chap. 3, Ex. 32 b)

$\boxed{b}$ Define

$$W_j^{(\nu)} = \frac{d^\nu}{dt^\nu} W_j(t)\Big|_{t=0} \quad (j=0,1,\ldots;\ \nu=0,1,\ldots). \tag{3}$$

In particular, $W_j^{(0)} = W_j(0) = 1$.

Suppose that $W_j(t)$ has the Maclaurin series representation

$$W_j(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} W_j^{(\nu)} \quad (j=0,1,\ldots). \tag{4}$$

According to Exercise 30,

$$P\{W>t \mid W>0\} = (1-\rho)\sum_{j=0}^{\infty} \rho^j W_j(t)$$

By (4),

$$\begin{aligned} P\{W>t \mid W>0\} &= (1-\rho)\sum_{j=0}^{\infty} \rho^j \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} W_j^{(\nu)} \\ &= (1-\rho)\sum_{j=0}^{\infty} \rho^j \Big[1 + \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} W_j^{(\nu)}\Big] \end{aligned}$$

Use of $\Sigma_{j=0}^{\infty} \rho^j = (1-\rho)^{-1}$, and a change of the order of summation yield

$$P\{W>t \mid W>0\} = 1 + (1-\rho)\sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} \sum_{j=0}^{\infty} \rho^j W_j^{(\nu)}. \tag{5}$$

$\boxed{c}$ Repeated differentiation of Equation (2) gives

$$\begin{aligned} \frac{d^\nu}{dt^\nu} W_j(t) = {} & \lambda \frac{d^{\nu-1}}{dt^{\nu-1}} W_{j+1}(t) + \frac{j}{j+1} s\mu \frac{d^{\nu-1}}{dt^{\nu-1}} W_{j-1}(t) \\ & - (\lambda+s\mu) \frac{d^{\nu-1}}{dt^{\nu-1}} W_j(t) \qquad [j=0,1,\ldots,\ \nu=1,2,\ldots] \end{aligned}$$

Setting $t=0$ we obtain

$$W_0^{(\nu)} = \lambda W_1^{(\nu-1)} - (\lambda+s\mu) W_0^{(\nu-1)} \qquad (\nu=1,2,\ldots),$$

$$W_j^{(\nu)} = \lambda W_{j+1}^{(\nu-1)} + \frac{j}{j+1} s\mu W_{j-1}^{(\nu-1)} - (\lambda+s\mu) W_j^{(\nu-1)} \qquad \begin{pmatrix} j=1,2,\ldots; \\ \nu=1,2,\ldots \end{pmatrix}.$$

First we solve for $\nu=1$. Recalling that $W_j^{(0)}=1$ for all j, we easily derive

$$W_j^{(1)} = -\frac{s\mu}{j+1} \qquad (j=0,1,\ldots). \tag{$*$}$$

(Chap. 3, Ex. 32 c)

Next we solve for $\nu = 2$, making use of (*). The result is

$$W_j^{(2)} = \begin{cases} (s\mu)^2[1+\frac{\rho}{2}] & (j=0), \\ (s\mu)^2 \frac{\rho}{(j+1)(j+2)} & (j \geq 1). \end{cases} \qquad (**)$$

By (*),

$$\sum_{j=0}^{\infty} \rho^j W_j^{(1)} = -s\mu \sum_{j=0}^{\infty} \frac{\rho^j}{j+1} = -s\mu \frac{1}{\rho} \sum_{j=1}^{\infty} \frac{\rho^j}{j}.$$

For $0<\rho<1$, $\Sigma_{j=1}^{\infty} \rho^j/j = -\ln(1-\rho) = \ln\frac{1}{1-\rho}$. Hence,

$$\sum_{j=0}^{\infty} \rho^j W_j^{(1)} = -s\mu \frac{1}{\rho} \ln \frac{1}{1-\rho}.$$

By (**),

$$\sum_{j=0}^{\infty} \rho^j W_j^{(2)} = (s\mu)^2 [1 + \sum_{j=0}^{\infty} \frac{\rho^{j+1}}{(j+1)(j+2)}] = (s\mu)^2[1 + \frac{1}{\rho} \sum_{j=1}^{\infty} \frac{\rho^{j+1}}{j(j+1)}].$$

Now let

$$S(\rho) = \sum_{j=1}^{\infty} \frac{\rho^{j+1}}{j(j+1)}.$$

Considering $S(\rho)$ as a function of ρ, differentiation results in $dS(\rho)/d\rho = \Sigma_{j=1}^{\infty} \rho^j/j = -\ln(1-\rho)$. Therefore, reversing the process, $S(\rho) = c + \int(-\ln(1-\rho))d\rho = c + (1-\rho)\ln(1-\rho) - (1-\rho)$. From $S(0) = 0$ we derive $c = 1$. Thus

$$\sum_{j=1}^{\infty} \frac{\rho^{j+1}}{j(j+1)} = \rho - (1-\rho) \ln \frac{1}{1-\rho}.$$

it follows that

$$\sum_{j=0}^{\infty} \rho^j W_j^{(2)} = (s\mu)^2 [2 - \frac{1-\rho}{\rho} \ln \frac{1}{1-\rho}].$$

Finally, substitution of the found expressions for $\Sigma_{j=0}^{\infty} \rho^j W_j^{(1)}$ and $\Sigma_{j=0}^{\infty} \rho^j W_j^{(2)}$ into Equation (5) gives

$$P\{W>t|W>0\} = 1 - s\mu t \frac{1-\rho}{\rho} \ln \frac{1}{1-\rho} + \frac{(s\mu t)^2}{2!}(1-\rho)[2 - \frac{1-\rho}{\rho} \ln \frac{1}{1-\rho}] - + \cdots \qquad (6)$$

□

Chapter 3, Exercise 33

'Let $B(t)$ be the distribution function of the busy period...'

It is clear that when service is in reverse order of arrival, then, for all $N \geq 1$, the waiting time is a busy period initiated by the presently served customer (whose remaining time in service is exponentially distributed) and including all arrivals later than the test customer until he is permitted to enter service. (The same holds true for a $GI/M/s$ system for $N \geq s$.) Hence, for an arbitrary customer, $P\{W \leq t \mid W > 0\} = B(t)$. It follows that the mean waiting time for waiting customers is the mean of the busy period, that is, $E(W \mid W > 0) = b = \tau/(1-\rho)$, by (4.12).

Chapter 3, Exercise 34

'Show that $\lim \Pi_j[n] = P_j, \ldots$'

By (7.7) and $\hat{a} = \gamma/\mu$,

$$\Pi_j[n] = \frac{\binom{n-1}{j}\left(\frac{\gamma}{\mu}\right)^j}{\sum_{k=0}^{s}\binom{n-1}{k}\left(\frac{\gamma}{\mu}\right)^k} \qquad (j = 0, 1, \ldots, s).$$

For $j = 0, 1, \ldots, s$,

$$\lim_{\substack{n\to\infty \\ \gamma\to 0 \\ n\gamma=\lambda}} \binom{n-1}{j}\left(\frac{\gamma}{\mu}\right)^j = \lim_{n\to\infty} \binom{n-1}{j}\left(\frac{\lambda}{\mu n}\right)^j$$

$$= \left[(\lambda/\mu)^j/j!\right] \lim_{n\to\infty}\left(\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{n-j}{n}\right)$$

$$= (\lambda/\mu)^j/j!$$

Hence,

$$\lim_{\substack{n\to\infty \\ \gamma\to 0 \\ n\gamma=\lambda}} \Pi_j[n] = \frac{\lim\binom{n-1}{j}\left(\frac{\gamma}{\mu}\right)^j}{\sum_{k=0}^{s}\lim\binom{n-1}{k}\left(\frac{\gamma}{\mu}\right)^k} = \frac{(\lambda/\mu)^j/j!}{\sum_{k=0}^{s}(\lambda/\mu)^k/k!} = P_j,$$

where, by Equation (3.3), P_j is the statistical equilibrium probability of j busy servers in the Erlang loss system. □

Chapter 3, Exercise 35

'Four sources share access to two servers.'

n = number of sources; $s = 2$; $\gamma^{-1} = 27$ min.; $\mu^{-1} = 3$ min.; $\hat{a} = \gamma/\mu = \frac{1}{9}$; blocked customers cleared.

The blocking probability is given by Engset's formula, that is Eq. (7.7) for $j = s$:

$$\Pi_s[n] = \frac{\binom{n-1}{s}\hat{a}^s}{\sum_{k=0}^{s}\binom{n-1}{k}\hat{a}^k}.$$

Since

$$\Pi_2[4] = \frac{3(\frac{1}{9})^2}{1 + 3(\frac{1}{9}) + 3(\frac{1}{9})^2} = \frac{1}{37} = 0.0270,$$

and

$$\Pi_2[5] = \frac{6(\frac{1}{9})^2}{1 + 4(\frac{1}{9}) + 6(\frac{1}{9})^2} = \frac{2}{41} = 0.0488,$$

the effect of going from four to five sources is a percent increase in the probability of blocking equal to

$$P = 100\,[(\Pi_2[5]/\Pi_2[4]) - 1] = 80\,\%.$$

To calculate the expected number of requests for service per hour, say r, we go through the following steps:

(i) $P_s[n] = \binom{n}{s}\hat{a}^s / \sum_{k=0}^{s}\binom{n}{k}\hat{a}^k$ (7.3),

(ii) $a' = a^*(1 - (1 - \frac{s}{n})P_s[n])$, where $a^* = n\hat{a}/(1+\hat{a})$ (7.8),

(iii) $a = a'/(1 - \Pi_s[n])$ (7.9),

(iv) $r = 60 \cdot a/\mu^{-1} = 20 \cdot a$.

For $n = 4$ and $n = 5$ we find

	$P_2[n]$	a'	a	r
$n = 4$	$\frac{2}{41} = 0.0488$	$\frac{16}{41} = 0.3902$	$\frac{148}{369} = 0.4011$	8.02
$n = 5$	$\frac{5}{68} = 0.0735$	$\frac{65}{136} = 0.4779$	$\frac{205}{408} = 0.5025$	10.05

The lower bounds for r are 8 and 10, respectively. □

Chapter 3, Exercise 36

'Verify equation (8.10)'

By Equations (6.8), (8.3), (8.6), (8.7) and (8.8),

$$\begin{aligned}
P\{W>t\} &= \sum_{j=0}^{n-s-1} P\{W>t \mid N=s+j\}\, P\{N=s+j\} \\
&= \sum_{j=0}^{n-s-1}\Big(e^{-s\mu t}\sum_{i=0}^{j}\frac{(s\mu t)^i}{i!}\Big) P_{s+j}[n-1] \\
&= e^{-s\mu t}\sum_{j=0}^{n-s-1}\Big(\sum_{i=0}^{j}\frac{(s\mu t)^i}{i!}\Big)\frac{(n-1)!\,\hat{a}^{s+j}}{(n-1-s-j)!\,s!\,s^j}\, P_0[n-1] \\
&= c\, e^{-\phi(t)}\sum_{j=0}^{n-s-1}\sum_{i=0}^{j}\frac{(\frac{\hat{a}}{s})^{-(n-1-s-j)}}{(n-1-s-j)!}\,\frac{(s\mu t)^i}{i!},
\end{aligned}$$

where

$$\phi(t) = \frac{s\mu}{\gamma} + s\mu t$$

and

$$c = \pi_0[n]\,\frac{(n-1)!\,\hat{a}^s}{s!}\Big(\frac{\hat{a}}{s}\Big)^{n-s-1} e^{s\mu/\gamma}$$

Thus

$$P\{W>t\} = c\,e^{-\phi(t)}\sum_{j=0}^{n-s-1}\sum_{i=0}^{j}\frac{(s\mu/\gamma)^{n-1-s-j}}{(n-1-s-j)!}\,\frac{(s\mu t)^i}{i!}.$$

By the substitution $k = n-1-s-j$,

$$P\{W>t\} = c\,e^{-\phi(t)}\sum_{k=0}^{n-s-1}\sum_{i=0}^{n-s-1-k}\frac{(s\mu/\gamma)^k}{k!}\,\frac{(s\mu t)^i}{i!}.$$

Defining $x = s\mu/\gamma$ and $y = s\mu t$ the doublesum may be written

$$\sum_{k=0}^{n-s-1}\sum_{i=0}^{n-s-1-k}\frac{x^k}{k!}\frac{y^i}{i!} = \frac{x^0y^0}{0!\,0!} + \Big(\frac{x^0y^1}{0!\,1!}+\frac{x^1y^0}{1!\,0!}\Big) + \Big(\frac{x^0y^2}{0!\,2!}+\frac{x^1y^1}{1!\,1!}+\frac{x^2y^0}{2!\,0!}\Big) + \cdots + \Big(\frac{x^0y^{n-s-1}}{0!\,(n-s-1)!}+\cdots+\frac{x^{n-s-1}y^0}{(n-s-1)!\,0!}\Big)$$

By the binomial formula, $\sum_{m=0}^{j}\frac{x^m}{m!}\frac{y^{j-m}}{(j-m)!} = \frac{(x+y)^j}{j!}$. Hence

$$\sum_{k=0}^{n-s-1}\sum_{i=0}^{n-s-1-k}\frac{x^k}{k!}\frac{y^i}{i!} = \sum_{j=0}^{n-s-1}\frac{(x+y)^j}{j!}.$$

We conclude that

$$P\{W>t\} = c\sum_{j=0}^{n-s-1}\frac{[\phi(t)]^j}{j!}\,e^{-\phi(t)} \qquad (8.10) \qquad \square$$

Chapter 3, Exercise 37

'Verify equation (8.18) by direct calculation'

As $a' = a$ by (8.16), then

$$\sum_{j=1}^{n} j P_j[n] = \Big(\sum_{j=1}^{s} j P_j[n] + \sum_{j=s+1}^{n} s P_j[n]\Big) + \sum_{j=s+1}^{n} (j-s) P_j[n]$$
$$= a + \sum_{k=0}^{n-s-1} (k+1) P_{s+k+1}[n] \qquad (1)$$

By (8.3),

$$\frac{P_{s+k+1}[n]}{P_{s+k}[n-1]} = \frac{n\hat{a}}{s} \frac{P_0[n]}{P_0[n-1]}. \qquad (2)$$

By (1) and (2),

$$\sum_{j=1}^{n} j P_j[n] = a + n\hat{a} \frac{P_0[n]}{P_0[n-1]} \frac{1}{\mu^{-1}} \sum_{k=0}^{n-s-1} \frac{k+1}{s\mu} P_{s+k}[n-1]. \qquad (3)$$

$P_{s+k}[n-1] = P\{N = s+k\}$ by (8.7) and $(k+1)/s\mu = E(W \mid N = s+k)$. Substitution into (3) and application of $E(W) = \sum_{k=0}^{n-s-1} E(W \mid N = s+k) P\{N = s+k\}$, see (8.13), gives

$$\sum_{j=1}^{n} j P_j[n] = a + n\hat{a} \frac{P_0[n]}{P_0[n-1]} \frac{E(W)}{\mu^{-1}}. \qquad (4)$$

By (8.4),

$$\frac{P_0[n]}{P_0[n-1]} = \sum_{j=0}^{s-1} \binom{n-1}{j} \hat{a}^j P_0[n] + \sum_{j=s}^{n-1} \frac{(n-1)!}{(n-1-j)!\, s!\, s^{j-s}} \hat{a}^j P_0[n]$$
$$= \frac{1}{n} \Big(\sum_{j=0}^{s-1} (n-j) \binom{n}{j} \hat{a}^j P_0[n] + \sum_{j=s}^{n} (n-j) \frac{n!}{(n-j)!\, s!\, s^{j-s}} \hat{a}^j P_0[n] \Big).$$

By (8.3) this is seen to equal

$$\frac{P_0[n]}{P_0[n-1]} = \frac{1}{n} \sum_{j=0}^{n} (n-j) P_j[n] = \frac{1}{n} \Big(n - \sum_{j=0}^{n} j P_j[n]\Big)$$

Substituting $n - \sum_{j=0}^{n} j P_j[n] = a/\hat{a}$ from (8.17) we finally obtain

$$\frac{P_0[n]}{P_0[n-1]} = \frac{a}{n\hat{a}}. \qquad (5)$$

By (4) and (5),

$$\sum_{j=1}^{n} j P_j[n] = a\Big(1 + \frac{E(W)}{\mu^{-1}}\Big) \qquad (8.18)$$

□

Chapter 3, Exercise 38

'Reconsider Exercise 35, but instead of...'

n = number of sources, $s = 2$, $\gamma^{-1} = 27$ min., $\mu^{-1} = 3$ min., $\hat{a} = \gamma/\mu = \frac{1}{9}$; blocked customers delayed.

To begin, we calculate the state distribution $\{P_j[n]\}_j$, for $n = 3, 4, 5$, using Equations (8.3) and (8.4).

$P_j[n]$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$n=3$	.7286	.2429	.0270	.0015	–	–
$n=4$	.6548	.2910	.0485	.0054	.0003	–
$n=5$	.5875	.3264	.0725	.0121	.0013	.0001

[a] Blocking probability.

$$P_B = \sum_{j=s}^{n-1} \Pi_j[n] = \sum_{j=s}^{n-1} P_j[n-1].$$

$n = 4$: $\quad P_B = \sum_{j=2}^{3} P_j[3] = 0.0270 + 0.0015 = \underline{0.0285}$

$n = 5$: $\quad P_B = \sum_{j=2}^{4} P_j[4] = 0.0485 + 0.0054 + 0.0003 = \underline{0.0542}$

The percent increase in blocking probability is

$$P = 100\,[(\sum_{j=2}^{4} \Pi_j[5] / \sum_{j=2}^{3} \Pi_j[4]) - 1] = 90\,\%$$

[b] Requests for service per hour.

$$r = 60\,a/\mu^{-1} = 20\,a.$$

Since $a = a'$ for a BCD system,

$$a = a' = \sum_{j=0}^{s-1} j P_j[n] + s \sum_{j=s}^{n} P_j[n] = P_1[n] + 2(1 - P_0[n] - P_1[n]) = 2 - 2P_0[n] - P_1[n].$$

$n = 4$: $\quad a = a' = 2 - 2 \cdot 0.6548 - 0.2910 = 0.3994,$

$\quad r = 20a = \underline{7.988}$

$n = 5$: $\quad a = a' = 2 - 2 \cdot 0.5875 - 0.3264 = 0.4986,$

$\quad r = 20a = \underline{9.972}$

Upper bounds for r are 8 and 10, respectively.

(Chap. 3, Ex 38 c)

$\boxed{c}$ Server occupancy.

$$\rho = a'/s .$$

$n = 4$: $\rho = 0.3994/2 = \underline{0.1997}$

$n = 5$: $\rho = 0.4986/2 = \underline{0.2493}$

$\boxed{d}$ Mean waiting time

By Eqs. (8.7), (8.9) and (8.13), the mean waiting time in seconds is

$$E(W) = \frac{60}{s\mu} \sum_{i=s}^{n-1} (i-s+1)\, P_i[n-1] .$$

$n = 4$: $E(W) = 90(P_2[3] + 2P_3[3]) = 90(0.0270 + 2\cdot 0.0015) = \underline{2.7}$ sec

$n = 5$: $E(W) = 90(P_2[4] + 2P_3[4] + 3P_4[4]) = 90(0.0485 + 2\cdot 0.0054 + 3\cdot 0.0003) = \underline{5.4}$ sec

$\boxed{e}$ $P\{W > 45 \text{ sec.}\}$

By Eqs. (8.7), (8.10), (8.11) and (8.12),

$$P\{W > t\} = c_0 e^{-s\mu t} \sum_{j=0}^{n-s-1} \frac{[\phi(t)]^j}{j!} , \qquad (8.10a)$$

where

$$\phi(t) = \frac{s\mu}{\gamma} + s\mu t \qquad (8.11)$$

and

$$c_0 = P_0[n-1] \frac{(n-1)!\, \hat{a}^s}{s!} \left(\frac{\hat{a}}{s}\right)^{n-s-1} , \qquad (8.12a)$$

with t measured in minutes.

For $t = 3/4$ min., $e^{-s\mu t} = e^{-1/2} = 0.6065$ and $\phi(t) = 18.5$. Also, $\hat{a} = 1/9$ and $\hat{a}/s = 1/18$.

$n = 4$: $c_0 = P_0[3] \dfrac{3!}{2!\,9^2} \dfrac{1}{18} = \dfrac{0.7286}{486} = 0.001499,$

$P\{W > 3/4\} = c_0 e^{-1/2}(1 + 18.5) = 0.001499 \cdot 0.6065 \cdot 19.5 = \underline{0.0177}.$

$n = 5$: $c_0 = P_0[4] \dfrac{4!}{2!\,9^2} \dfrac{1}{18^2} = \dfrac{0.6548}{2187} = 0.000299,$

$P\{W > 3/4\} = c_0 e^{-1/2}(1 + 18.5 + \frac{18.5^2}{2!}) = 0.000299 \cdot 0.6065 \cdot 190.6 = \underline{0.0346}.$

(Chap. 3, Ex. 38 f)

$\boxed{f}$ Proportion of time a source is idle

Evidently,

$$f = \frac{\gamma^{-1}}{\gamma^{-1} + E(W)/60 + \mu^{-1}} \qquad \left(= \frac{n - \sum_{j=1}^{n} j P_j[n]}{n} \right).$$

$n = 4$: $\quad f = 27/(27 + 2.7/60 + 3) = \underline{0.899}$.

$n = 5$: $\quad f = 27/(27 + 5.4/60 + 3) = \underline{0.897}$.

Compare with upper bound 0.9.

Chapter 3, Exercise 39

'Using Equations (6.3) and (8.17), show that $a\Pi_j[n] = (n-j)\hat{a}P_j[n]$...'

Assume a BCD system with quasi-random input generated by n sources and with exponential service times. By Eq. (6.3),

$$\Pi_j[n] = \frac{(n-j)\,P_j[n]}{\sum_{k=0}^{n-1}(n-k)\,P_k[n]} \qquad (j = 0, 1, \ldots, n-1). \qquad (6.3)$$

Clearly, $\Pi_n[n] = 0$. Thus (6.3) is valid also for $j = n$. Furthermore, extending the summation to include $k = n$ does not affect the value of the denominator. Hence, by (6.3),

$$\Pi_j[n] = \frac{(n-j)\,P_j[n]}{n - \sum_{k=1}^{n} k P_k[n]} \qquad (j = 0, 1, \ldots, n). \qquad (*)$$

Now, by Eq. (8.17),

$$n - \sum_{k=1}^{n} k P_k[n] = \frac{a}{\hat{a}}.$$

Inserting this expression into (*) we obtain

$$a\,\Pi_j[n] = (n-j)\,\hat{a}\,P_j[n] \qquad (j = 0, 1, \ldots, n). \qquad \square$$

Chapter 3, Exercise 40

'Consider a single-server queueing system with quasirandom...'

The queueing process under consideration is a birth-and-death process with $\lambda_j = (n-j)\gamma$ and $\mu_j = j\mu$ for $j = 0,1,\ldots,n$. Use of Eq. (1.1) results in the equilibrium state probabilities

$$P_j[n] = \binom{n}{j}\hat{a}^j P_0[n] \qquad (j = 0,1,\ldots,n),$$

with $\hat{a} = \gamma/\mu$. As $1 = \Sigma_{j=0}^{n} P_j[n] = (1+\hat{a})^n P_0[n]$,

$$P_j[n] = \frac{\binom{n}{j}\hat{a}^j}{(1+\hat{a})^n} \qquad (j = 0,1,\ldots,n). \qquad (*)$$

Since we deal with a queue with quasirandom input and blocked customers delayed, it is true that $\Pi_j[n] = P_j[n-1]$ for all $j = 0,1,\ldots,n-1$. Hence, by (*),

$$\Pi_j[n] = \frac{\binom{n-1}{j}\hat{a}^j}{(1+\hat{a})^{n-1}} \qquad (j = 0,1,\ldots,n-1).$$

Chapter 3, Exercise 41

'Queue with feedback'.

The arrival rate of new customers to the system is λ. The effective departure rate (from system) per customer in service is $(1-p)\mu$. Thus the queueing process is a birth-and-death process with the parameters $\lambda_n = \lambda$ for all n, and $\mu_n = n(1-p)\mu$ for $0 \leq n \leq s-1$, $\mu_n = s(1-p)\mu$ for $n \geq s$. Offered load is $a = \lambda/[(1-p)\mu]$.

The state of the system behaves precisely as in an ordinary BCD queue with parameters s, λ and $(1-p)\mu$. Also, $\Pi_j = P_j$ due to Poisson arrivals, where $\{\Pi_j\}$ is the arrival distribution for new customers. The equilibrium probability that a new arrival finds all servers busy equals

$$C(s,a) = \sum_{j=s}^{\infty} \Pi_j = \sum_{j=s}^{\infty} P_j \qquad (a < s),$$

with $C(s,a)$ given by Erlang's delay formula, Eq. (4.8). □

Chapter 3, Exercise 42

'A single server serves customers of two priority classes...'

Poisson arrivals and exponential service times are assumed for both customer classes. The parameters are λ_1 and μ_1 for the high priority class, λ_2 and μ_2 for the low priority class. In parts (a)-(e) preemptive-repeat priority discipline will be assumed.

[a] By Eq. (5.23) of Chapter 2, the probability of preemption for a class 2 customer who has just entered or reentered service equals $\lambda_1/(\lambda_1+\mu_2)$. Hence, the number N of preemptions experienced by a class 2 customer has the geometric distribution

$$P\{N=k\} = \left(\frac{\lambda_1}{\lambda_1+\mu_2}\right)^k \frac{\mu_2}{\lambda_1+\mu_2} \quad (k=0,1,\ldots) \qquad (1)$$

[b] The accumulated service time of a class 2 customer is not affected by preemptions (which in effect only interrupt the service), given exponential service time and preemptive-repeat rule. Letting S denote the total time an arbitrary class 2 customer occupies the server, we have

$$P\{S \leqq t\} = 1 - e^{-\mu_2 t}, \qquad (2)$$

just as if there were no preemptions allowed.

[c] Let T denote the extended service time, composed of the actual service time S and the sum $\Sigma_{j=1}^{N} X_j$ of the N time intervals during which the customer is preempted from service:

$$T = S + \sum_{j=1}^{N} X_j . \qquad (3)$$

Since N and $\{X_j\}$ are independent, by part (b) of Exercise 4 of Chapter 2,

$$E(T) = E(S) + E(N)E(X), \qquad (4)$$

(Chap. 3, Ex. 42 c)

where E(X) denotes the common mean of $X_1, X_2, \ldots$. Now,

$$E(S) = \frac{1}{\mu_2}, \tag{5}$$

$$E(N) = \frac{\lambda_1/(\lambda_1+\mu_2)}{\mu_2/(\lambda_1+\mu_2)} = \frac{\lambda_1}{\mu_2}, \tag{6}$$

$$E(X) = \frac{\mu_1^{-1}}{1-(\lambda_1/\mu_1)} = \frac{1}{\mu_1-\lambda_1}. \quad (\mu_1 > \lambda_1). \tag{7}$$

Eq. (5) follows from Eq. (2). Eq. (6) follows from Eq. (1) since a variable with the geometric distribution $P\{N=k\} = q^k p$ has the mean q/p. See also Chapter 2, Exercise 21 a ($E(M) = \lambda\tau$). Eq. (7) follows from the observation that each X_j is a busy period in a single-server queue with only class 1 customers. Thus Eq. (4.12) applies with $\tau = \mu_1^{-1}$ and $a = \lambda_1/\mu_1$.

Substitution of (5), (6) and (7) into (4) yields

$$E(T) = \frac{\mu_1}{\mu_2(\mu_1-\lambda_1)} = \frac{\mu_2^{-1}}{1-a_1} \quad (\mu_1 > \lambda_1). \tag{8}$$

$\boxed{d}$ The service of high-priority customers is in no way affected by the presence of low-priority customers. Therefore, the waiting time W_1 of an arbitrary class 1 customer will have the distribution given by Eq. (4.25). Hence, by Exercise 4 g of Chapter 1,

$$P\{W_1 > t\} = C(1, \tfrac{\lambda_1}{\mu_1})\, e^{-(\mu_1-\lambda_1)t} = \frac{\lambda_1}{\mu_1} e^{-(\mu_1-\lambda_1)t} \tag{9}$$

$\boxed{e}$ Conditions for bounded delays:

$$\begin{aligned} &\text{High-priority customers}: \quad \frac{\lambda_1}{\mu_1} < 1, \\ &\text{Low-priority customers}: \quad \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} < 1 \end{aligned} \tag{10}$$

$\boxed{f}$ Under the exponential service time assumption, the remaining service time at preemption will be exponentially distributed with mean μ_2^{-1}. Hence, an assumption of preemptive-resume priority discipline does not change the results in parts a – e.

□

Chapter 3, Exercise 43

'Priority reservation.'

Let λ_E = eastbound traffic call rate, λ_W = westbound traffic call rate, μ = service rate. Hence the offered loads are $a_1 = \lambda_E/\mu$ and $a_2 = \lambda_W/\mu$, respectively.

$\boxed{a}$ Suppose $1 \leqq n \leqq s-1$. A westbound call will be cleared in arrival state $j \geq s-n$, whereas an eastbound call will be cleared only if $j = s$.

The queueing process can be modeled as a birth-and-death process with $\lambda_j = \lambda_E + \lambda_W$ for $j = 0, 1, \ldots, s-n-1$; $\lambda_j = \lambda_E$ for $j = s-n, \ldots, s-1$; $\mu_j = j\mu$ for $j = 0, 1, \ldots, s$. By Eq. (3.15) of Chapter 2 then

$$
\begin{aligned}
(\lambda_E + \lambda_W) P_0 &= 1 \cdot \mu P_1 \\
&\vdots \\
(\lambda_E + \lambda_W) P_{s-n-1} &= (s-n)\mu P_{s-n} \\
\lambda_E P_{s-n} &= (s-n+1)\mu P_{s-n+1} \\
&\vdots \\
\lambda_E P_{s-1} &= s\mu P_s .
\end{aligned}
$$

$\boxed{b}$ Recursive solution of the above state equations give

$$
P_j = \begin{cases} \dfrac{(a_1+a_2)^j}{j!} P_0 & (j = 1, 2, \ldots, s-n), \\[2ex] \left(\dfrac{a_1+a_2}{a_1}\right)^{s-n} \dfrac{a_1^j}{j!} P_0 & (j = s-n+1, \ldots, s). \end{cases}
$$

As usual, P_0 is found by use of the condition $\Sigma_{j=0}^{s} P_j = 1$.

$\boxed{c}$ Loss on eastbound traffic $= P_s$.

Loss on westbound traffic $= \sum_{j=s-n}^{s} P_j$. □

Chapter 3, Exercise 44

In order to minimize its telephone bill ...

[a] Equilibrium state probabilities for flat-rate queue.

The equilibrium state probabilities $\{P_j\}$ for the flat-rate queueing system can be found from the following equilibrium state equations:

$$
\begin{aligned}
(\lambda_1+\lambda_2) P_0 &= 1\cdot\mu P_1 \\
&\vdots \\
(\lambda_1+\lambda_2) P_{s-1} &= s\mu P_s \\
\lambda_2 P_s &= s\mu P_{s+1} \\
&\vdots \\
\lambda_2 P_{s+i} &= s\mu P_{s+i+1} \\
&\vdots
\end{aligned}
$$

By recursive solution,

$$
P_j = \begin{cases} \dfrac{(a_1+a_2)^j}{j!} P_0 & (j = 1, 2, \ldots, s-1), \\[2ex] \dfrac{(a_1+a_2)^s}{s!}\left(\dfrac{a_2}{s}\right)^{j-s} P_0 & (j = s, s+1, \ldots), \end{cases}
$$

and

$$
P_0 = \left[\sum_{k=0}^{s-1} \frac{(a_1+a_2)^k}{k!} + \frac{(a_1+a_2)^s}{s!} \frac{1}{1-a_2/s}\right]^{-1},
$$

with $a_1=\lambda_1/\mu$ and $a_2=\lambda_2/\mu$, where $a_2<s$. If $a_2 \geq s$, then $P_j=0$ for all j.

[b] The blocking probability $B(s) = \sum_{j=s}^{\infty} \Pi_j = \sum_{j=s}^{\infty} P_j$.

$$
B(s) = \frac{\dfrac{(a_1+a_2)^s}{s!} \dfrac{1}{1-a_2/s}}{\displaystyle\sum_{k=0}^{s-1} \frac{(a_1+a_2)^k}{k!} + \frac{(a_1+a_2)^s}{s!}\cdot\frac{1}{1-a_2/s}} \qquad (a_2 < s).
$$

Observe that calculation of $B(s)$ is facilitated by the formula

$$
B(s) = s\,B(s,a_1+a_2)/(s-a_2(1-B(s,a_1+a_2))),
$$

as is easily verified, and the recurrence of Exercise 6 of Chapter 3.

$\boxed{c}$ The overall cost per minute, $c(s)$

Cost parameters:
c = cost per minute of a flat-rate trunk,
r_0 = cost of a toll call for the first minute or fraction thereof,
r = cost of a toll call for each additional minute or fraction thereof.
Letting M denote the random number of 1-minute intervals beyond the initial 1-minute interval, obviously

$$c(s) = cs + \lambda_1 B(s)[r_0 + E(M)r],$$

since $\lambda_1 B(s)$ is the average overflow rate of high-priority customers requesting service from the flat-rate trunks, and $r_0 + E(M)r$ is the mean cost of a toll call.

Now, given exponential service time with mean μ^{-1}, the probability of holding the line for at least 1 more minute equals $e^{-\mu}$ at the start of each 1-minute interval. Therefore, $P\{M=k\} = (e^{-\mu})^k(1-e^{-\mu})$ for $k = 0, 1, \ldots$. Hence $E(M) = e^{-\mu}/(1-e^{-\mu})$, and

$$c(s) = cs + \lambda_1 [r_0 + \frac{r}{e^{\mu}-1}] B(s).$$

$\boxed{d}$ Mean waiting time for low-priority calls, $E(W_2)$

Let W_2 = waiting time of an arbitrary low-priority customer. Observe, $P_{s+j} = B(s)(a_2/s)^j(1-a_2/s)$ for $j = 0, 1, \ldots$, and, for N equal to the arrival state of the customer, $E(W_2|N=s+j) = (j+1)(s\mu)^{-1}$. Hence, for $a_2 < s$, since $\Pi_{s+j} = P_{s+j}$,

$$E(W_2) = \sum_{j=0}^{\infty} E(W_2|N=s+j)\Pi_{s+j} = B(s)(s\mu)^{-1}\sum_{j=0}^{\infty}(j+1)(\frac{a_2}{s})^j(1-\frac{a_2}{s})$$

$$= B(s)(s\mu)^{-1}(1-\frac{a_2}{s})^{-1} = \frac{B(s)}{s\mu-\lambda_2} \quad \text{[analogous with Eq. (4.27)]}$$

$\boxed{e}$ Occupancy of flat-rate trunks, ρ.

Clearly, $\rho = 1$ if $a_2 \geq s$. In case $a_2 < s$, the carried load on the flat-rate server group is $a' = a_1[1-B(s)] + a_2$. Hence,

$$\rho = \frac{a'}{s} = \frac{a_1[1-B(s)]+a_2}{s} \qquad (a_2 < s).$$

$\square$

Chapter 3, Exercise 45

'Time-varying Poisson input'

$\boxed{a}$ First assume that $\lambda(t)$ is continuous and differentiable for all t. By the reasoning used for derivation of Eq. (2.5) of Chapter 2 we find

$$\frac{d}{dt} P_j(t) = \lambda(t) P_{j-1}(t) - \lambda(t) P_j(t) \qquad [j = 0,1,\ldots; P_{-1}(t) = 0],$$

the initial condition being $P_0(0) = 1$. Solution by recurrence starting with $j = 0$ yields the Poisson distribution

$$P_j(t) = \frac{(\Lambda(t))^j}{j!} e^{-\Lambda(t)} \qquad (j = 0,1,..) \qquad (1)$$

where

$$\Lambda(t) = \int_0^t \lambda(x)\,dx. \qquad (2)$$

The equations can be shown to hold also in the case of a piecewise continuous and differentiable $\lambda(t)$. This may be done by utilizing the additivity property of the Poisson distribution.

$\boxed{b}$ In the infinite server queue a customer who arrives at time $x < t$ will still be in service at time t with probability $1 - H(t-x)$. Hence, counting only arrivals that will be in the system at time t, the effective arrival rate at $x < t$ equals $\lambda(x) = \lambda[1 - H(t-x)]$. The corresponding counting process is a Poisson process with time-varying rate. By (1) and (2) the number of customers in the system (= in service) at t will have the Poisson distribution with mean

$$\Lambda(t) = \int_0^t \lambda[1-H(t-x)]\,dx = \lambda \int_0^t [1-H(x)]\,dx$$

$$= \lambda[t(1-H(t)) + \int_0^t x\,dH(x)]$$

$$= \lambda t\, p(t),$$

where $p(t) = 1 - H(t) + \int_0^t \frac{x}{t}\,dH(x)$. This proves Equation (3.11).

Finally, we observe that also Eq. (4.26) of Chapter 2 may be proved in a similar way by appeal to the notion of a time-varying Poisson process. $\square$

$\boxed{\text{Chapter 3, Exercise 46}}$

'Transient analysis of the single-server Erlang delay model.'

$\boxed{a}$ For the M/M/1 system, clearly

$$P_0(t+h) = P_0(t)[1-\lambda h] + P_1(t)\mu h + o(h),$$

$$P_j(t+h) = P_{j-1}(t)\lambda h + P_j(t)[1-(\lambda+\mu)h] + P_{j+1}(t)\mu h + o(h) \qquad (j=1,2,\dots).$$

Hence,

$$\frac{d}{dt}P_0(t) = -\lambda P_0(t) + \mu P_1(t),$$

$$\frac{d}{dt}P_j(t) = \lambda P_{j-1}(t) - (\lambda+\mu)P_j(t) + \mu P_{j+1}(t) \qquad (j=1,2,\dots).$$

Choosing μ^{-1} as the time unit, then $\mu = 1$ and $\lambda = \lambda/\mu = a$, so that the above equation system becomes

$$\frac{d}{dt}P_0(t) = -a P_0(t) + P_1(t), \tag{1}$$

$$\frac{d}{dt}P_j(t) = a P_{j-1}(t) - (1+a)P_j(t) + P_{j+1}(t) \quad (j=1,2,\dots). \tag{2}$$

$\boxed{b}$ Consider the auxiliary system of equations

$$\frac{d}{dt}\hat{P}_j(t) = a\hat{P}_{j-1}(t) - (1+a)\hat{P}_j(t) + \hat{P}_{j+1}(t) \quad (j=0,\pm1,\pm2,\dots), \tag{3}$$

$$\hat{P}_0(t) = a\hat{P}_{-1}(t). \tag{4}$$

(3) and (4) together imply

$$\frac{d}{dt}\hat{P}_0(t) = -a\hat{P}_0(t) + \hat{P}_1(t), \tag{1a}$$

and by (3),

$$\frac{d}{dt}\hat{P}_j(t) = a\hat{P}_{j-1}(t) - (1+a)\hat{P}_j(t) + \hat{P}_{j+1}(t) \quad (j=1,2,\dots) \tag{2a}$$

Thus, if $\hat{P}_j(t)$ $(j=0,\pm1,\pm2,\dots)$ is a solution to Eqs. (3) and (4), then $\hat{P}_j(t)$ $(j=0,1,2,\dots)$ will be a solution to Equations (1a) and (2a). As (1a) and (2a) are formally identical to (1) and (2), we conclude that if $\hat{P}_j(t)$ for $j=0,\pm1,\pm2,\dots$ solves (3) and (4), then $P_j(t) = \hat{P}_j(t)$, for $j=0,1,2,\dots$ and all t, will also solve (1) and (2).

(Chap. 3, Ex. 46 c)

$\boxed{c}$ Let $\hat{P}(z,t)$ denote the generating function

$$\hat{P}(z,t) = \sum_{j=-\infty}^{\infty} \hat{P}_j(t)\, z^j. \tag{6}$$

Multiplication of Eq. (3) by z^j and summation for all j result in

$$\sum_{j=-\infty}^{\infty} \frac{d}{dt}\hat{P}_j(t)\, z^j = az\sum_{j=-\infty}^{\infty} \hat{P}_{j-1}(t) z^{j-1} - (1+a)\sum_{j=-\infty}^{\infty} \hat{P}_j(t) z^j + z^{-1}\sum_{j=-\infty}^{\infty} \hat{P}_{j+1}(t) z^{j+1},$$

or,

$$\frac{d}{dt}\sum_{j=-\infty}^{\infty} \hat{P}_j(t) z^j = [az - (1+a) + z^{-1}]\sum_{j=-\infty}^{\infty} \hat{P}_j(t) z^j,$$

which, by (6), is the same as

$$\frac{d}{dt}\hat{P}(z,t) = [az - (1+a) + z^{-1}]\,\hat{P}(z,t), \tag{7}$$

whose general solution is

$$\hat{P}(z,t) = G(z)\, e^{[-(1+a)t + (az+z^{-1})t]}, \tag{8}$$

where $G(z)$ is any function of z.

$\boxed{d}$ Eq. (8) may be rewritten as

$$\hat{P}(z,t) = G(z)\, e^{-(1+a)t}\, e^{\frac{1}{2}[2a^{1/2}t][(a^{1/2}z)+(a^{1/2}z)^{-1}]}.$$

We shall use the fact that

$$e^{\frac{1}{2}y(x+x^{-1})} = \sum_{k=-\infty}^{\infty} I_k(y)\, x^k, \tag{9}$$

where $I_k(y)$ are the modified Bessel functions. Now, setting $y = 2a^{1/2}t$ and $x = a^{1/2}z$ it is seen immediately that

$$\hat{P}(z,t) = G(z)\, e^{-(1+a)t}\sum_{k=-\infty}^{\infty} I_k(2a^{1/2}t)\, a^{k/2} z^k. \tag{10}$$

(Chap. 3, Ex. 46 e)

$\boxed{e}$ Suppose $G(z)$ has the expansion

$$G(z) = \sum_{j=-\infty}^{\infty} c_{-j}\, z^j. \tag{11}$$

Insertion into (10) and collection of terms by powers of z lead to

$$\hat{P}(z,t) = \sum_{j=-\infty}^{\infty} \Big(e^{-(1+a)t} \sum_{k=-\infty}^{\infty} c_k a^{\frac{1}{2}(j+k)} I_{j+k}(2a^{1/2}t) \Big) z^j.$$

Comparison with Eq. (6) shows that

$$\hat{P}_j(t) = e^{-(1+a)t} \sum_{k=-\infty}^{\infty} c_k a^{\frac{1}{2}(j+k)} I_{j+k}(2a^{1/2}t). \tag{12}$$

$\boxed{f}$ For $y=0$ Eq. (9) specializes to

$$1 = \sum_{k=-\infty}^{\infty} I_k(0)\, x^k,$$

whereby $I_0(0) = 1$ and $I_k(0) = 0$ for $k \neq 0$. By (12) then

$$\hat{P}_j(0) = \sum_{k=-\infty}^{\infty} c_k a^{\frac{1}{2}(j+k)} I_{j+k}(0) = c_{-j}.$$

Let i = initial state, so that $P_i(0) = 1$. Then (provided $\hat{P}_j(t) = P_j(t)$ for $j = 0, 1, \ldots$) $c_{-i} = 1$, and $c_k = 0$ for $k \leq 0$ but $k \neq -i$. Thus Eq. (12) can be written

$$\hat{P}_j(t) = e^{-(1+a)t}\Big[a^{\frac{1}{2}(j-i)} I_{j-i}(2a^{1/2}t) + \sum_{k=1}^{\infty} c_k a^{\frac{1}{2}(j+k)} I_{j+k}(2a^{1/2}t)\Big]. \tag{13}$$

$\boxed{g}$ By Eq. (13),

$$\hat{P}_0(t) = e^{-(1+a)t}\Big[a^{-\frac{1}{2}i} I_{-i}(2a^{1/2}t) + \sum_{k=1}^{\infty} c_k a^{\frac{1}{2}k} I_k(2a^{1/2}t)\Big],$$

$$\hat{P}_{-1}(t) = e^{-(1+a)t}\Big[a^{-\frac{1}{2}-\frac{1}{2}i} I_{-(i+1)}(2a^{1/2}t) + \sum_{k=1}^{\infty} c_k a^{-\frac{1}{2}+\frac{1}{2}k} I_{k-1}(2a^{1/2}t)\Big]$$

From these expressions and Eq. (4), $\hat{P}_0(t) = a\hat{P}_{-1}(t)$, we obtain

$$a^{-\frac{1}{2}i} I_{-i} + \sum_{k=1}^{\infty} d_k I_k = a^{-\frac{1}{2}i+\frac{1}{2}} I_{-(i+1)} + a^{1/2} \sum_{k=1}^{\infty} d_k I_{k-1}, \tag{14}$$

where $d_k = c_k a^{\frac{1}{2}k}$ and $I_k = I_k(2a^{1/2}t)$.

(Chap. 3, Ex. 46 h)

$\boxed{h}$ Since $I_{-k}(y) = I_k(y)$, Eq. (14) can be written

$$a^{-\frac{1}{2}i} I_i + \sum_{k=1}^{\infty} d_k I_k = a^{-\frac{1}{2}i+\frac{1}{2}} I_{i+1} + a^{\frac{1}{2}} \sum_{k=1}^{\infty} d_k I_{k-1} .$$

This equation should hold for all t and hence for all arguments $2a^{1/2}t$ of the I_r's $(r = 0, 1, \ldots)$. Consequently, the coefficient of each I_r must equal zero. For example, for $i=2$ this requirement leads to the following set of equations,

$$\begin{aligned}
0 &= a^{\frac{1}{2}} d_1 \\
d_1 &= a^{\frac{1}{2}} d_2 \\
a^{-\frac{1}{2}i} + d_2 &= a^{\frac{1}{2}} d_3 \\
d_3 &= a^{\frac{1}{2}} d_4 + a^{-\frac{1}{2}i+\frac{1}{2}} \\
d_4 &= a^{\frac{1}{2}} d_5 \\
d_5 &= a^{\frac{1}{2}} d_6 \\
&\vdots
\end{aligned}$$

For arbitrary initial state i the solution is

$$\begin{aligned}
d_k &= 0 \qquad (k = 1, 2, \ldots, i) \qquad [\text{void if } i = 0] \\
d_{i+1} &= a^{-\frac{1}{2}i - \frac{1}{2}} \\
d_{i+1+m} &= a^{-\frac{1}{2}i - \frac{1}{2}m - \frac{1}{2}} (1-a) \qquad (m = 1, 2, \ldots)
\end{aligned}$$

$\boxed{i}$ Substituting the found values of $d_k = c_k a^{\frac{1}{2}k}$ into (13), at the same time replacing $\tilde{P}_j(t)$ with $P_j(t)$ (the initialization permits this), we get

$$P_j(t) = e^{-(1+a)t} \Big[a^{\frac{1}{2}(j-i)} I_{j-i}(2a^{1/2}t) + a^{\frac{1}{2}(j-i)-\frac{1}{2}} I_{j+i+1}(2a^{1/2}t) + (1-a) \sum_{m=1}^{\infty} a^{\frac{1}{2}(j-i)-\frac{1}{2}m-\frac{1}{2}} I_{j+i+1+m}(2a^{1/2}t) \Big] .$$

Simplification results in

$$P_j(t) = a^{\frac{1}{2}(j-i)} e^{-(1+a)t} \Big[I_{j-i}(2a^{1/2}t) + a^{-\frac{1}{2}} I_{j+i+1}(2a^{1/2}t) + (1-a) \sum_{k=2}^{\infty} a^{-k/2} I_{j+i+k}(2a^{1/2}t) \Big] . \tag{15}$$

$\boxed{j}$ Equation (15) holds for all values of the carried load a. $\square$