Asymptotic Analysis on an Unbounded Zero-One Knapsack with Discrete-Sized Objects*

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Abstract

The problem being discussed in this paper is a special case of the unbounded knapsack problem:

$$\max z_n(M) = \frac{1}{n} \sum_{i=1}^{n} p_i x_i$$

s.t.  $\sum_{i=1}^{n} c_i x_i \leq \beta_n n$

where $p_i$’s are uniformly distributed random variables in $[0, 1]$, and $c_i$’s are discrete random variables distributed uniformly in $\{1/M, 2/M, \ldots, (M-1)/M, 1\}$. Assuming that $M$ is large, it is shown that $\lim_{n \to \infty} z_n(M)$ approximately equals to $\sqrt{\frac{2\ln n}{n}} - 0.3062(\sqrt{\frac{\ln n}{n}})^{-1}$.

1 Introduction

The unbounded knapsack problem can be stated as the following combinatorial optimization problem:

$$\max z_n(M) = \frac{1}{n} \sum_{i=1}^{n} p_i x_i$$

s.t.  $\sum_{i=1}^{n} c_i x_i \leq \beta_n n$

where $p_i$’s are uniformly distributed random variables in $[0, 1]$, and $c_i$’s are discrete random variables distributed uniformly in $\{1/M, 2/M, \ldots, (M-1)/M, 1\}$.

First, the average profit gain for the case that $c_i$’s are continuous will be derived. Assuming that each bidder whose original bid size is in $[j/M, (j+1)/M]$ will bid for either $j/M$ or $(j+1)/M$ equally random, the average profit gain can be derived.

The rest of the paper will elucidate the derivation and an application in multi-unit combinatorial auction. The average profit gain for the case that $c_i$’s are continuous random variables will be derived in the next section. The average profit gain for discrete $c_i$’s will be derived in Section 3. Section 4 presents an application of the result in multi-unit combinatorial auction. Finally, the conclusion will be presented in Section 5.

2 Average profit gain: Continuous $c_i$’s

For the case that $c_i$’s are continuous random variables distributed uniformly in $[0, 1]$, the optimal solution $\lim_{n \to \infty} z_n$ can be obtained by simply solving the second equality constraint in (1) for $m^*$ and, then, substituting the value to the first equality for the optimal $\lim_{n \to \infty} z_n$. Solving the second equality in (1), we are

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*The research is supported in part by the National High Technology Research and Development Program (863 Program) No. 2001AA113180 and a grant from the Departmental Earning Account, Department of Computing, Hong Kong Polytechnic University.
able to show that

\[ m^* = \sqrt{\frac{1}{6\beta_0}} \quad (3) \]

Putting this result back to the first equality in (1), we have

\[ \lim_{n \to \infty} z_n = \sqrt{\frac{2\beta_0}{3}} \quad (4) \]

This result provides the researchers the first approximation on the average profit being gained from each object. But the assumption is that the product is divisible, i.e., the size of each object can be in any fractional number.

3 Average profit gain: Discrete \( c_i \)'s

Let us consider a simple multi-unit combinatorial auction problem. Suppose an auctioneer has 1 000 000 units of notebook computers for auction off. The auctioneer anticipates that 2 000 bidders will come. Moreover, the auctioneer also expects that their bid sizes are uniformly random in \([1, 1000]\) and the bid price is another independent random variables uniformly in \([100, 1000]\). Accordingly, the average profit gain is estimated as follows:

\[ 2 000 \times 100 000 \times 5 \times 2000 \approx 2 000 \times 100 000 \times \sqrt{\frac{2000}{3}} \approx 115.47 \times 10^6. \]

Since the quantized levels in both bid size and bid price are small, 0.001 and 0.001 respectively, the approximation is close to the actual value. However, it will be inaccurate if the quantized level is not small.

Let \( M \) be the maximum size that a bidder will bid. The actual inequality constraint will be expressed as follows:

\[ \frac{1}{n} \sum_{i=1}^{n} c_i M x_i \leq \beta_0 M. \quad (5) \]

Here \( c_i M \in \{1, 2, \ldots, M\} \), is the actual bid size for the \( i^{th} \) bidder. Let \( \lim_{n \to \infty} z_n (M) \) be the optimal average profit gain.

\[ \lim_{n \to \infty} z_n (M) \leq \lim_{n \to \infty} z_n. \]

Equality holds when \( M \to \infty \). Besides, let us define the percentage error, \( E(M) \), as follows:

\[ E(M) = \lim_{n \to \infty} \frac{\sqrt{\beta_0 M}}{\sqrt{\lim_{n \to \infty} z_n}} \]

We now consider the case that the probability distribution for the bid size \( c \), \( P(c) \), is uniformly distributed in the set \( \{1/M, 2/M, \ldots, 1\} \), that is,

\[ P(c) = \frac{1}{M} \delta(c, k/M) \quad (7) \]

\[ \delta(c, k/M) = \begin{cases} 1 & \text{if } c = k/M \\ 0 & \text{otherwise.} \end{cases} \]

This result provides the researchers the first approximation on the average profit being gained from each object. But the assumption is that the product is divisible, i.e., the size of each object can be in any fractional number.

Figure 1: The straight line \( p = m^* c \) is the decision boundary for product allocation. All the bids \((p, c)\) above this line will be allocated.

In practice, we assume that each bidder whose original bid size is in the range \([j/M, (j+1)/M]\) \((j = 0, 1, \ldots, M - 1)\) will bid up to \((j+1)/M\). Therefore, the equality constraint in Equation (1) can be written as follows:

\[ \sum_{i=1}^{k^*} \left( 1 - \frac{m^*}{M} \right) \left( \frac{i}{M} \right)^2 = \beta_0 \quad (8) \]

where \( m^* \) is the slope of the straight line for the decision boundary, see Figure 1, and \( m^* \) and \( k^* \) can be related by the following inequality.

\[ k^* \frac{M}{m^*} < \frac{1}{m^*} < k^* + 1 \frac{M}{m^*}. \]

Solving Equation (8), it is able to obtain

\[ k^* \frac{(k^* + 1)}{2} = m^* \frac{k^* (k^* + 1) (2k^* + 1)}{6} = \beta_0 M^2. \quad (9) \]

Since the number of bidders \( n \) and the number of quantized intervals \( M \) are large, it can be assumed that \( k^* \gg 1 \) and \( m^* / M \approx k^* \). Hence,

\[ k^* \approx \sqrt{6\beta_0 M}. \quad (10) \]

The average profit gain \( \lim_{n \to \infty} z_n (M) \) can be written as follows:

\[ \lim_{n \to \infty} z_n (M) = \frac{1}{M} \int_{k/M}^{1} p dp = \frac{1}{2M} k^* - \frac{6M}{12M} k^* - \frac{3}{12M} - \frac{1}{12k^* M} \]

\[ \approx \frac{1}{3M} k^* - \frac{1}{4M}. \]
Using Equation (11), we can determine the maximum integer of $M$ such that the percentage error is less than a certain threshold $\eta$.

$$M \approx \frac{0.3062}{\eta} \sqrt{\frac{2\beta_0}{\beta_0}}.$$ 

For $\eta = 0.01$ and $\beta_0 = 0.2$, it can readily be shown that $M \approx 69$. For $\eta = 0.01$ and $\beta_0 = 0.5$, $M \approx 44$. For $M$ equals to 100 and $\beta_0$ is in $[0.2, 0.5]$, the percentage error is less than 1%.

5 Conclusion

The average profit gain, Equation (4), for the unbounded knapsack problem with fractional sized objects has been derived. An extended result for the case that object size is discrete has been obtained. It has been shown that the average discrete-sized profit gain can be expressed in terms of $\beta_0$ and $M$ as $\sqrt{\frac{2\beta_0}{\beta_0}} - 0.3062(\sqrt{\beta_0} M)^{-1}$. An application of this result in multi-unit combinatorial auction has been presented. Without loss of generality, the case when $y_i$'s are discrete random variables can also be derived using the same technique used in the paper.

References


4 Application in MUCA

In the illustrative example mentioned in the last section, we roughly estimated the average profit gain, lim_{N \to \infty} \lim_{M \to \infty} z(M)$, as $\sqrt{\frac{2(0.5)}{3}}$. In accordance with Equation (11), the percentage error can thus be obtained.

$$E(M) = 0.3062 \frac{1}{\sqrt{0.5}} \frac{1}{1000} = 4.33 \times 10^{-4}$$

which is less than 0.05%. A better estimation for the average profit gain can also be evaluated as follows:

$$\lim_{M \to 1000} z(M) = (1 - 4.33 \times 10^{-4}) \sqrt{\frac{2(0.5)}{3}} = 0.5771$$